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The analysis of thermal residual stresses near the apex in bonded dissimilar materials

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Abstract

By employing the complex variable method and constructing the particular solution sequences in the form of complex functions, all the cases of the thermal residual stress field near the apex in dissimilar materials bonded with two arbitrary angles are researched theoretically, and the corresponding classical solutions are obtained. Moreover, the primary paradox, the secondary paradox and even the triple paradox are discovered in the classical solutions and also resolved here, thereby it is confirmed that thermal residual stresses near the apex in bonded dissimilar materials probably possess the singularities of $\ln r$ (when the primary paradox occurs), $\ln^2 r$ (when the secondary paradox occurs) and even $\ln^3 r$ (when the triple paradox occurs). In addition, the systematic method to solve multiple paradox problems is put forward.

1. Introduction

For the structures jointed with different isotropic homogeneous elastic materials, thermal residual stresses develop during the cooling-down process in manufacture due to the difference in thermal expansion coefficients of these materials. These stresses may lower the strength of bonded dissimilar materials, and sometimes give rise to damage in them, it is therefore necessary to investigate theoretically the distribution and the singularities of thermal residual stresses near the apex in bonded dissimilar materials. Mizuno et al. (1988) and Suga et al. (1989) studied thermal stresses in two dissimilar materials jointed together with right angles (90°/90°), and came to the conclusion that the order of singularity developing under thermal stress loading is the same as that under mechanical loading. However, by applying the boundary element method and analyzing numerical results, Yuuki and Xu (1992) and Yuuki et al. (1991) found that logarithmic singularities may develop at the apex under thermal stress loading. The preliminary theoretical research on

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thermal residual stresses near the apex in bonded dissimilar materials made by Xu and Mutoh (1996) confirmed the above discovery, but they did not make further investigations so as to clarify the whole situation of the problem. In addition, for two dissimilar materials jointed together with right angles or with two arbitrary angles $\theta_1(>0)$ and $\theta_2(<0)$, Munz and Yang (1992) and Munz et al. (1993) discovered that the thermoelastic constant stress terms σ_{ij0} (i.e., the classical solutions of thermal stresses) approach infinity for a combination of the elastic constants leading to a stress singularity exponent $\omega = 0$, this is a paradox. For the special case of a free straight surface, i.e., $\theta_1 - \theta_2 = 180°$, Ioka et al. (1996) also discovered that the paradox exists for the thermoelastic constant stress terms σ_{ij0}, they pointed out that logarithmic free edge stress singularity appears for thermal residual stresses when the paradox occurs, and demonstrated numerically the conclusion by using the boundary element method.

In this study, we employ the complex variable method, by constructing the particular solution sequences in the form of complex functions (Ding et al., 1998), all the cases of the thermal residual stress field near the apex in dissimilar materials bonded with two arbitrary angles are researched theoretically, and the corresponding classical solutions are presented. Moreover, the primary paradox, the secondary paradox and even the triple paradox are discovered in the classical solutions, the solutions for the paradox are also obtained here, from which it is confirmed that thermal residual stresses near the apex in bonded dissimilar materials probably possess the singularities of $\ln r$ (when the primary paradox occurs), $\ln^2 r$ (when the secondary paradox occurs) and even $\ln^3 r$ (when the triple paradox occurs). The discovery of the triple paradox is for the first time, and the systematic method to solve multiple paradox problems is also put forward in the paper.

2. Model of the problem and basic equations

The model of dissimilar materials jointed with two arbitrary angles θ_1 and θ_2 is shown in Fig. 1, then the boundary conditions are

$$\text{At}\quad \theta = \theta_j:\quad \sigma_{j\theta}(r, \theta_j) = 0, \quad \tau_{jr\theta}(r, \theta_j) = 0 \quad (j = 1, 2) \tag{1}$$

where the subscript j stands for the two materials. Supposing that the structure is cooled down

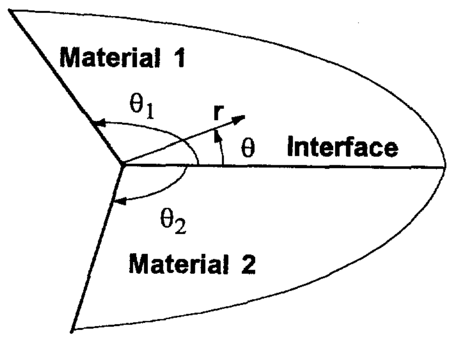

Fig. 1. Dissimilar materials jointed with two arbitrary angles θ_1 and θ_2.

from a stress-free state, the resulting temperature difference is ΔT, which is defined as negative for cooling conditions, the thermal expansion coefficient for material j is $\alpha_j (j = 1,2)$, thus the continuous conditions for the stresses and displacements at the interface are given by

$$\text{at} \quad \theta = 0: \quad \sigma_{1\theta}(r,0) = \sigma_{2\theta}(r,0), \quad \tau_{1r\theta}(r,0) = \tau_{2r\theta}(r,0) \tag{2a}$$

$$u_{1r}(r,0) = u_{2r}(r,0) + (\alpha_2^* - \alpha_1^*)\Delta T \cdot r, \quad u_{1\theta}(r,0) = u_{2\theta}(r,0) \tag{2b}$$

where

$$\alpha_j^* = \begin{cases} (1+v_j)\alpha_j & \text{for plane strain} \\ \alpha_j & \text{for plane stress} \end{cases} \tag{3}$$

$v_j (j = 1,2)$ being the Poisson's ratio of material j, $u_{jr}(r,\theta)$ and $u_{j\theta}(r,\theta)$ are the displacements caused by thermal residual stresses $\sigma_{jr}(r,\theta)$, $\sigma_{j\theta}(r,\theta)$ and $\tau_{jr\theta}(r,\theta)$.

Following the complex function theory of Muskhelishvili (1953), we can write the displacements and stresses for each material, in term of two complex potentials $\varphi_j(z)$ and $\psi_j(z)$ $(j = 1,2)$ as

$$\sigma_{jr} - i\tau_{jr\theta} = \varphi_j'(z) + \overline{\varphi_j'(z)} - e^{2i\theta}[\bar{z}\varphi_j''(z) + \psi_j'(z)] \tag{4}$$

$$\sigma_{j\theta} + i\tau_{jr\theta} = \varphi_j'(z) + \overline{\varphi_j'(z)} + e^{2i\theta}[\bar{z}\varphi_j'' + \psi_j'(z)] \tag{5}$$

$$2\mu_j(u_{jr} + iu_{j\theta}) = e^{-i\theta}[\kappa_j\varphi_j(z) - \overline{z\varphi_j'(z)} - \overline{\psi_j(z)}] \tag{6}$$

where

$$\kappa_j = \begin{cases} 3 - 4v_j & \text{for plane strain} \\ \dfrac{3 - v_j}{1 + v_j} & \text{for plane stress} \end{cases} \tag{7}$$

$\mu_j (j = 1,2)$ being the shear modulus of material j, $\overline{\varphi_j(z)}$ and $\overline{\psi_j(z)}$ are the complex conjugates of $\varphi_j(z)$ and $\psi_j(z)$ (where $z = re^{i\theta}$), respectively.

We assume

$$\left.\begin{aligned} \varphi_j(z) &= z(A_j + C_j \ln z + E_j \ln^2 z + G_j \ln^3 z + M_j \ln^4 z) \\ \psi_j(z) &= z(B_j + D_j \ln z + F_j \ln^2 z + H_j \ln^3 z + N_j \ln^4 z) \end{aligned}\right\} \quad (j = 1,2) \tag{8}$$

where $A_j, B_j, \ldots, M_j$ and N_j are complex constants.

Substituting (8) into (5) and (6), one obtains

$$\begin{aligned} \sigma_{j\theta} + i\tau_{jr\theta} = {} & (M_j + \overline{M_j} + e^{2i\theta}N_j)\ln^4 r \\ & + [G_j + \overline{G_j} + 8M_j + 4\overline{M_j} + 4i\theta(M_j - \overline{M_j} + e^{2i\theta}N_j) + e^{2i\theta}(H_j + 4N_j)]\ln^3 r \\ & + \{E_j + \overline{E_j} + 6G_j + 3\overline{G_j} + 12M_j + 3i\theta[G_j - \overline{G_j} + 8M_j - 4\overline{M_j} + e^{2i\theta}(H_j + 4N_j)] \\ & - 6\theta^2(M_j + \overline{M_j} + e^{2i\theta}N_j) + e^{2i\theta}(F_j + 3H_j)\}\ln^2 r + \{C_j + \overline{C_j} + 4E_j + 2\overline{E_j} + 6G_j \\ & + 2i\theta[E_j - \overline{E_j} + 6G_j - 3\overline{G_j} + 12M_j + e^{2i\theta}(F_j + 3H_j)] - 3\theta^2[G_j + \overline{G_j} + 8M_j + 4\overline{M_j} \\ & + e^{2i\theta}(H_j + 4N_j)] - 4i\theta^3(M_j - \overline{M_j} + e^{2i\theta}N_j) + e^{2i\theta}(D_j + 2F_j)\}\ln r + A_j + \overline{A_j} + 2C_j \end{aligned}$$

$$+\overline{C_j}+2E_j+i\theta[C_j-\overline{C_j}+4E_j-2\overline{E_j}+6G_j+e^{2i\theta}(D_j+2F_j)]-\theta^2[E_j+\overline{E_j}+6G_j+3\overline{G_j}+12M_j+e^{2i\theta}(F_j+3H_j)]-i\theta^3[G_j-\overline{G_j}+8M_j-4\overline{M_j}+e^{2i\theta}(H_j+4N_j)]+\theta^4(M_j+\overline{M_j}+e^{2i\theta}N_j)+e^{2i\theta}(B_j+D_j) \tag{9}$$

$$\begin{aligned}2\mu_j(u_{jr}+iu_{j\theta}) &= (\kappa_j M_j-\overline{M_j}-e^{-2i\theta}\overline{N_j})r\ln^4 r\\ &+[\kappa_j G_j-\overline{G_j}-4\overline{M_j}-e^{-2i\theta}\overline{H_j}+4i\theta(\kappa_j M_j+\overline{M_j}+e^{-2i\theta}\overline{N_j})]r\ln^3 r\\ &+[\kappa_j E_j-\overline{E_j}-3\overline{G_j}-e^{-2i\theta}\overline{F_j}+3i\theta(\kappa_j G_j+\overline{G_j}+4\overline{M_j}+e^{-2i\theta}\overline{H_j})-6\theta^2(\kappa_j M_j-\overline{M_j}\\ &-e^{-2i\theta}\overline{N_j})]r\ln^2 r+[\kappa_j C_j-\overline{C_j}-2\overline{E_j}-e^{-2i\theta}\overline{D_j}+2i\theta(\kappa_j E_j+\overline{E_j}+3\overline{G_j}+e^{-2i\theta}\overline{F_j})\\ &-3\theta^2(\kappa_j G_j-\overline{G_j}-4\overline{M_j}-e^{-2i\theta}\overline{H_j})-4i\theta^3(\kappa_j M_j+\overline{M_j}+e^{-2i\theta}\overline{N_j})]r\ln r+[\kappa_j A_j\\ &-\overline{A_j}-\overline{C_j}-e^{-2i\theta}\overline{B_j}+i\theta(\kappa_j C_j+\overline{C_j}+2\overline{E_j}+e^{-2i\theta}\overline{D_j})-\theta^2(\kappa_j E_j-\overline{E_j}-3\overline{G_j}\\ &-e^{-2i\theta}\overline{F_j})-i\theta^3(\kappa_j G_j+\overline{G_j}+4\overline{M_j}+e^{-2i\theta}\overline{H_j})+\theta^4(\kappa_j M_j-\overline{M_j}-e^{-2i\theta}\overline{N_j})]r\end{aligned} \tag{10}$$

The applications of the boundary conditions (1) and the interface conditions (2a), (2b) to (9) and (10) yield

$$\left.\begin{aligned}N_j &= -(M_j+\overline{M_j})\,e^{-2i\theta_j}\\ H_j &= -(G_j+\overline{G_j}+4M_j-8i\theta_j\overline{M_j})\,e^{-2i\theta_j}\\ F_j &= -(E_j+\overline{E_j}+3G_j-6i\theta_j\overline{G_j}-24\theta_j^2\overline{M_j})\,e^{-2i\theta_j}\\ D_j &= -(C_j+\overline{C_j}+2E_j-4i\theta_j\overline{E_j}-12\theta_j^2\overline{G_j}+32i\theta_j^3\overline{M_j})\,e^{-2i\theta_j}\\ B_j &= -(A_j+\overline{A_j}+C_j-2i\theta_j\overline{C_j}-4\theta_j^2\overline{E_j}+8i\theta_j^3\overline{G_j}+16\theta_j^4\overline{M_j})\,e^{-2i\theta_j}\end{aligned}\right\}\quad (j=1,2) \tag{11}$$

$$(M_1+\overline{M_1})(1-e^{2i\theta_1})=(M_2+\overline{M_2})(1-e^{2i\theta_2}) \tag{12a}$$

$$(G_1+\overline{G_1}+4\overline{M_1})(1-e^{2i\theta_1})-8i\theta_1\,e^{2i\theta_1}M_1=(G_2+\overline{G_2}+4\overline{M_2})(1-e^{2i\theta_2})-8i\theta_2\,e^{2i\theta_2}M_2 \tag{12b}$$

$$(E_1+\overline{E_1}+3\overline{G_1})(1-e^{2i\theta_1})-(6i\theta_1 G_1-24\theta_1^2 M_1)\,e^{2i\theta_1}=(E_2+\overline{E_2}+3\overline{G_2})(1-e^{2i\theta_2})-(6i\theta_2 G_2-24\theta_2^2 M_2)\,e^{2i\theta_2} \tag{12c}$$

$$(C_1+\overline{C_1}+2\overline{E_1})(1-e^{2i\theta_1})-(4i\theta_1 E_1-12\theta_1^2 G_1-32i\theta_1^3 M_1)\,e^{2i\theta_1}=(C_2+\overline{C_2}+2\overline{E_2})(1-e^{2i\theta_2})-(4i\theta_2 E_2-12\theta_2^2 G_2-32i\theta_2^3 M_2)\,e^{2i\theta_2} \tag{12d}$$

$$(A_1+\overline{A_1}+\overline{C_1})(1-e^{2i\theta_1})-(2i\theta_1 C_1-4\theta_1^2 E_1-8i\theta_1^3 G_1+16\theta_1^4 M_1)\,e^{2i\theta_1}=(A_2+\overline{A_2}+\overline{C_2})(1-e^{2i\theta_2})-(2i\theta_2 C_2-4\theta_2^2 E_2-8i\theta_2^3 G_2+16\theta_2^4 M_2)\,e^{2i\theta_2} \tag{12e}$$

$$M_2=\Gamma\frac{\kappa_1+1}{\kappa_2+1}M_1+\frac{1-\Gamma}{\kappa_2+1}(M_1+\overline{M_1}(1-e^{2i\theta_1})$$

$$G_2 = \Gamma\frac{\kappa_1+1}{\kappa_2+1}G_1 + \frac{1-\Gamma}{\kappa_2+1}[(G_1+\overline{G_1}+4\overline{M_1})(1-e^{2i\theta_1})-8i\theta_1 e^{2i\theta_1}M_1]$$

$$E_2 = \Gamma\frac{\kappa_1+1}{\kappa_2+1}E_1 + \frac{1-\Gamma}{\kappa_2+1}[(E_1+\overline{E_1}+3\overline{G_1})(1-e^{2i\theta_1})-(6i\theta_1 G_1-24\theta_1^2 M_1)e^{2i\theta_1}]$$

$$C_2 = \Gamma\frac{\kappa_1+1}{\kappa_2+1}C_1 + \frac{1-\Gamma}{\kappa_2+1}[(C_1+\overline{C_1}\,2\overline{E_1})(1-e^{2i\theta_1})$$

$$-(4i\theta_1 E_1-12\theta_1^2 G_1-32i\theta_1^3 M_1)e^{2i\theta_1}]$$

$$A_2 = \Gamma\frac{\kappa_1+1}{\kappa_2+1}A_1 + \frac{1-\Gamma}{\kappa_2+1}[(A_1+\overline{A_1}+\overline{C_1})(1-e^{2i\theta_1})-(2i\theta_1 C_1-4\theta_1^2 E_1$$

$$-8i\theta_1^3 G_1+16\theta_1^4 M_1)e^{2i\theta_1}]-\frac{2\mu_2}{\kappa_2+1}(\alpha_2^*-\alpha_1^*)\Delta T \tag{13}$$

where $\Gamma = \mu_2/\mu_1$. Equations (11) are derived from the traction-free conditions (1); equations (12) from (2a), the stress continuous conditions on the interface $\theta = 0$, and with the applications of (11); equations (13) from (2b), the displacement continuous conditions on the interface $\theta = 0$, and with the applications of (11) and (12).

The procedure to solve the above basic equations is: getting first the constants M_j, G_j, E_j, C_j and A_j one by one from equations (12) and (13), then from eqns (11) the constants N_j, H_j, F_j, D_j and B_j can be obtained.

Noticing that eqn (12a) is actually a set of linear homogeneous equations about two real constants $M_1+\overline{M_1}$ and $M_2+\overline{M_2}$, the coefficient determinant of which is

$$\Delta = (1-\cos 2\theta_1)\sin 2\theta_2-(1-\cos 2\theta_2)\sin 2\theta_1 = 4\sin\theta_1\cdot\sin\theta_2\cdot\sin(\theta_1-\theta_2) \tag{14}$$

when $\Delta \neq 0$, $M_1+\overline{M_1} = 0$ and $M_2+\overline{M_2} = 0$, while when $\Delta = 0$, non-trivial solutions may exist for $M_1+\overline{M_1}$ and $M_2+\overline{M_2}$, hence we make discussions separately for the case of $\Delta \neq 0$ and of $\Delta = 0$ below.

3. The particular solutions when $\Delta \neq 0$

3.1. The classical solution

In this case we can take $E_j = F_j = G_j = H_j = M_j = N_j = 0$, $(j = 1, 2)$. Solving eqns (12) and (13), and from eqns (11) one obtains

$$C_1 = i\frac{1}{\Theta_1}\cdot\frac{4\mu_2}{\kappa_2+1}(\alpha_2^*-\alpha_1^*)\Delta T\cdot\Delta$$

$$\frac{A_1+\overline{A_1}}{2} = \frac{1}{\Theta_1}\cdot\frac{2\mu_2}{\kappa_2+1}(\alpha_2^*-\alpha_1^*)\Delta T\cdot[Q_1\sin 2\theta_2-Q_2(1-\cos 2\theta_2)]$$

$$
\begin{aligned}
C_2 &= i\frac{1}{\Theta_1}\cdot\frac{4\mu_2}{\kappa_2+1}(\alpha_2^*-\alpha_1^*)\Delta T\cdot\Gamma\frac{\kappa_1+1}{\kappa_2+1}\Delta \\
A_2 &= \Gamma\frac{\kappa_1+1}{\kappa_2+1}\cdot\frac{A_1-\overline{A_1}}{2}+\frac{1}{\Theta_1}\cdot\frac{2\mu_2}{\kappa_2+1}(\alpha_2^*-\alpha_1^*)\Delta T[Q_1\sin 2\theta_1-Q_2(1-\cos 2\theta_1)] \\
&\quad -i\frac{1}{\Theta_1}\cdot\frac{4\mu_2}{\kappa_2+1}(\alpha_2^*-\alpha_1^*)\Delta T\cdot\frac{1-\Gamma}{\kappa_2+1}\{[Q_1\sin 2\theta_2-Q_2(1-\cos 2\theta_2)]\sin 2\theta_1 \\
&\quad +\Delta(1-\cos 2\theta_1-2\theta_1\sin 2\theta_1)\} \\
D_1 &= D_2 = 0 \\
B_1 &= -\frac{1}{\Theta_1}\cdot\frac{4\mu_2}{\kappa_2+1}(\alpha_2^*-\alpha_1^*)\Delta T\{[Q_1\sin 2\theta_2-Q_2(1-\cos 2\theta_2)]+(i-2\theta_1)\Delta\}\,e^{-2i\theta_1} \\
B_2 &= -\frac{1}{\Theta_1}\cdot\frac{4\mu_2}{\kappa_2+1}(\alpha_2^*-\alpha_1^*)\Delta T \\
&\quad \cdot\left\{[Q_1\sin 2\theta_1-Q_2(1-\cos 2\theta_1)]+(i-2\theta_2)\Gamma\frac{\kappa_1+1}{\kappa_2+1}\Delta\right\}e^{-2i\theta_2}
\end{aligned}
\tag{15}
$$

where

$$
\begin{aligned}
Q_1 &= \Gamma\frac{\kappa_1+1}{\kappa_2+1}(-\sin 2\theta_2+2\theta_2\cos 2\theta_2)-(-\sin 2\theta_1+2\theta_1\cos 2\theta_1) \\
Q_2 &= \Gamma\frac{\kappa_1+1}{\kappa_2+1}(1-\cos 2\theta_2-2\theta_2\sin 2\theta_2)-(1-\cos 2\theta_1-2\theta_1\sin 2\theta_1)
\end{aligned}
\tag{16}
$$

$$
\begin{aligned}
\Theta_1 &= \left[\Gamma\frac{\kappa_1+1}{\kappa_2+1}+2\frac{1-\Gamma}{\kappa_2+1}(1-\cos 2\theta_1)\right]\cdot[Q_1\sin 2\theta_2-Q_2(1-\cos 2\theta_2)] \\
&\quad -[Q_1\sin 2\theta_1-Q_2(1-\cos 2\theta_1)]+2\frac{1-\Gamma}{\kappa_2+1}(-\sin 2\theta_1+2\theta_1\cos 2\theta_1)\Delta
\end{aligned}
\tag{17}
$$

then from (8) we have

$$
\left.\begin{aligned}
\varphi_1(z) &= \frac{A_1-\overline{A_1}}{2}z+\frac{A_1+\overline{A_1}}{2}z+C_1 z\ln z \\
\psi_1(z) &= B_1 z
\end{aligned}\right\}
\quad\text{and}\quad
\left.\begin{aligned}
\varphi_2(z) &= A_2 z+C_2 z\ln z \\
\psi_2(z) &= B_2 z
\end{aligned}\right\}
\tag{18}
$$

where $A_1-\overline{A_1}\ /2$ is an arbitrary imaginary constant, its related terms represent a rigid body rotation. Equation (18) is just the classical solution.

Substituting (18) into (4), (5) and noticing $C_j+\overline{C_j}=0$ yields

$$
\left.\begin{aligned}
\sigma_{jr}-i\tau_{jr\theta} &= [1+e^{2i(\theta-\theta_j)}]\,[A_j+\overline{A_j}+(1+2i\theta_j)C_j]+2i(\theta-\theta_j)C_j-2C_j \\
\sigma_{j\theta}+i\tau_{jr\theta} &= [1-e^{2i(\theta-\theta_j)}]\,[A_j+\overline{A_j}+(1+2i\theta_j)C_j]+2i(\theta-\theta_j)C_j
\end{aligned}\right\}\quad (j=1,2)
\tag{19}
$$

Equations (18) and (19) are valid only when $\Theta_1 \neq 0$, on this condition the expression (19), which is a particular solution of thermal residual stresses, does not contain $\ln r$ terms, namely, the logarithmic singularity will not appear.

Introducing the Dundurs' parameters (Dundurs, 1969)

$$\alpha = \frac{\Gamma(\kappa_1+1)-(\kappa_2+1)}{\Gamma(\kappa_1+1)+(\kappa_2+1)}, \quad \beta = \frac{\Gamma(\kappa_1-1)-(\kappa_2-1)}{\Gamma(\kappa_1+1)+(\kappa_2+1)} \tag{20a}$$

hence

$$\Gamma\frac{\kappa_1+1}{\kappa_2+1} = \frac{1+\alpha}{1-\alpha}, \quad \frac{1-\Gamma}{\kappa_2+1} = \frac{\beta-\alpha}{1-\alpha} \tag{20b}$$

we can rewrite eqn (17), the expression of Θ_1, as

$$\begin{aligned}\Theta_1 = &-\frac{2}{(1-\alpha)^2}\{(1+\alpha)^2(1-\cos 2\theta_2-\theta_2\sin 2\theta_2)+(1-\alpha)^2(1-\cos 2\theta_1-\theta_1\sin 2\theta_1)\\ &-(1+\alpha)(1-\alpha)\cdot[1-\cos 2\theta_1-\cos 2\theta_2+\cos 2(\theta_1-\theta_2)+(\theta_1-\theta_2)\sin 2(\theta_1-\theta_2)\\ &-\theta_1\sin 2\theta_1-\theta_2\sin 2\theta_2]+2(\beta-\alpha)[(1+\alpha)(1-\cos 2\theta_2-\theta_2\sin 2\theta_2)(1-\cos 2\theta_1)\\ &-(1-\alpha)(1-\cos 2\theta_1-\theta_1\sin 2\theta_1)(1-\cos\theta_2)]\}\end{aligned} \tag{21}$$

Analyses and calculations demonstrate that when $(\theta_1-\theta_2) > 45^\circ$ (for plain strain) the value of Θ_1 is zero for some special combinations of the geometry angles θ_1, θ_2 and the Dundurs' parameters α, β. Actually, by setting $\Theta_1 = 0$, we get

$$\begin{aligned}\beta = \alpha - &\{(1+\alpha)^2(1-\cos 2\theta_2-\theta_2\sin 2\theta_2)+(1-\alpha)^2(1-\cos 2\theta_1-\theta_1\sin 2\theta_1)\\ &-(1-\alpha^2)[1-\cos 2\theta_1-\cos 2\theta_2+\cos 2(\theta_1-\theta_2)+(\theta_1-\theta_2)\sin 2(\theta_1-\theta_2)\\ &-\theta_1\sin 2\theta_1-\theta_2\sin 2\theta_2]\}/\{2[(1+\alpha)(1-\cos 2\theta_2-\theta_2\sin 2\theta_2)(1-\cos 2\theta_1)\\ &-(1-\alpha)(1-\cos 2\theta_1-\theta_1\sin 2\theta_1)(1-\cos 2\theta_2)]\}\end{aligned} \tag{22}$$

Based on (22), some curves on which Θ_1 equals zero are shown in Fig. 2.

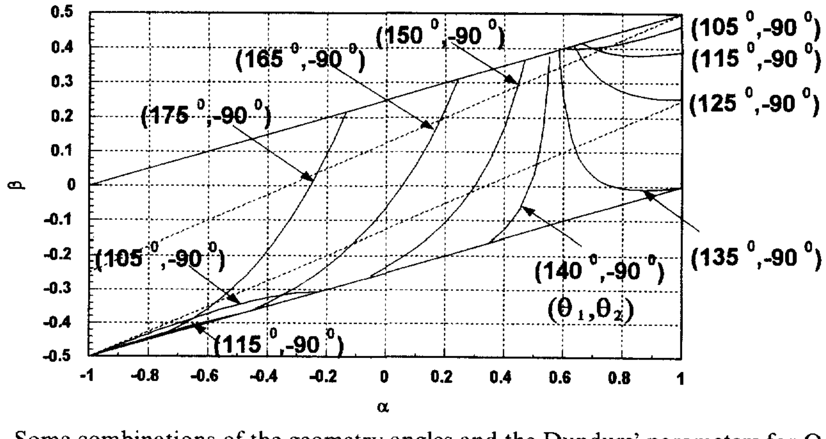

Fig. 2. Some combinations of the geometry angles and the Dundurs' parameters for $\Theta_1 = 0$.

When $\Theta_1 = 0$, the classical solution (18) becomes infinite, this is a paradox, and the expression (19), the stresses in which are called the regular stress terms by Munz et al. (1993), breaks down.

3.2. *The primary paradox solution (when $\Theta_1 = 0$)*

Taking $G_j = H_j = M_j = N_j = 0$, $(j = 1, 2)$. From eqns (12c) and (13) we have $E_j + \overline{E_j} = 0$ and

$$E_2 = \Gamma\frac{\kappa_1+1}{\kappa_2+1}E_1$$

$$C_2 = \Gamma\frac{\kappa_1+1}{\kappa_2+1}C_1 + \frac{1-\Gamma}{\kappa_2+1}[(C_1+\overline{C_1})(1-e^{2i\theta_1}) - 2E_1(1-e^{2i\theta_1}+2i\theta_1 e^{2i\theta_1})]$$

$$A_2 = \Gamma\frac{\kappa_1+1}{\kappa_2+1}A_1 + \frac{1-\Gamma}{\kappa_2+1}\left[(A_1+\overline{A_1})(1-e^{2i\theta_1}) - \frac{C_1-\overline{C_1}}{2}(1-e^{2i\theta_1}+2i\theta_1 e^{2i\theta_1})\right.$$
$$\left. + \frac{C_1+\overline{C_1}}{2}(1-e^{2i\theta_1}-2i\theta_1 e^{2i\theta_1}) + 4\theta_1^2 e^{2i\theta_1}E_1\right] - \frac{2\mu_2}{\kappa_2+1}(\alpha_2^*-\alpha_2^*)\Delta T \tag{23}$$

Substituting (23) into (12d) and (12e), then separating the real parts from the imaginary parts, we obtain

$$b_{11}\cdot(C_1+\overline{C_1}) + b_{12}\cdot 2E_1 i = 0 \tag{24a}$$

$$b_{21}\cdot(C_1+\overline{C_1}) + b_{22}\cdot 2E_1 i = 0 \tag{24b}$$

$$b_{11}\cdot(A_1+\overline{A_1}) + b_{12}\cdot\frac{C_1-\overline{C_1}}{2}i + b_{13}\cdot\frac{C_1+\overline{C_1}}{2} + b_{14}\cdot E_1 i$$
$$= -\frac{4\mu_2}{\kappa_2+1}(\alpha_2^*-\alpha_1^*)\Delta T(1-\cos 2\theta_2) \tag{25a}$$

$$b_{21}\cdot(A_1+\overline{A_1}) + b_{22}\cdot\frac{C_1-\overline{C_1}}{2}i + b_{23}\cdot\frac{C_1-\overline{C_1}}{2} + b_{24}\cdot E_1 i$$
$$= -\frac{4\mu_2}{\kappa_2+1}(\alpha_2^*-\alpha_1^*)\Delta T\sin 2\theta_2 \tag{25b}$$

where

$$b_{11} = (1-\cos 2\theta_1) - (1-\cos 2\theta_2)\left[\Gamma\frac{\kappa_1+1}{\kappa_2+1} + 2\frac{1-\Gamma}{\kappa_2+1}(1-\cos 2\theta_1)\right]$$

$$b_{21} = \sin 2\theta_1 - \sin 2\theta_2\left[\Gamma\frac{\kappa_1+1}{\kappa_2+1} + 2\frac{1-\Gamma}{\kappa_2+1}(1+\cos 2\theta_1)\right]$$

$$b_{12} = \Gamma\frac{\kappa_1+1}{\kappa_2+1}(-\sin 2\theta_2 + 2\theta_2\cos 2\theta_2) - (-\sin 2\theta_1 + 2\theta_1\cos 2\theta_1)$$

$$+2\frac{1-\Gamma}{\kappa_2+1}(-\sin 2\theta_1+2\theta_1\cos 2\theta_1)(1-\cos 2\theta_2)$$

$$b_{22}=\Gamma\frac{\kappa_1+1}{\kappa_2+1}(1-\cos 2\theta_2-2\theta_2\sin 2\theta_2)-(1-\cos 2\theta_1-2\theta_1\sin 2\theta_1)$$
$$+2\frac{1-\Gamma}{\kappa_2+1}(-\sin 2\theta_1+2\theta_1\cos 2\theta_1)\sin 2\theta_2$$

$$b_{13}=(1-\cos 2\theta_1+2\theta_1\sin 2\theta_1)-(1-\cos 2\theta_2+2\theta_2\sin 2\theta_2)\left[\Gamma\frac{\kappa_1+1}{\kappa_2+1}\right.$$
$$\left.+2\frac{1-\Gamma}{\kappa_2+1}(1-\cos 2\theta_1)\right]-2\frac{1-\Gamma}{\kappa_2+1}[(1-\cos 2\theta_1+2\theta_1\sin 2\theta_1)(1-\cos 2\theta_2)$$
$$-\sin 2\theta_1(-\sin 2\theta_2+2\theta_2\cos 2\theta_2)]$$

$$b_{23}=(\sin 2\theta_1+2\theta_1\cos 2\theta_1)-(\sin 2\theta_2+2\theta_2\cos 2\theta_2)\left[\Gamma\frac{\kappa_1+1}{\kappa_2+1}+2\frac{1-\Gamma}{\kappa_2+1}(1-\cos 2\theta_1)\right]$$
$$-2\frac{1-\Gamma}{\kappa_2+1}[1-\cos 2\theta_1+2\theta_1\sin 2\theta_1)\sin 2\theta_2-\sin 2\theta_1(1-\cos 2\theta_2-2\theta_2\sin 2\theta_2)]$$

$$b_{14}=4\theta_1^2\sin 2\theta_1-\Gamma\frac{\kappa_1+1}{\kappa_2+1}\cdot 4\theta_2^2\sin 2\theta_2-2\frac{1-\Gamma}{\kappa_2+1}[4\theta_1^2\sin 2\theta_1(1-\cos 2\theta_2)$$
$$+(1-\cos 2\theta_1-2\theta_1\sin 2\theta_1)(-\sin 2\theta_2+2\theta_2\cos 2\theta_2)$$
$$-(-\sin 2\theta_1+2\theta_1\cos 2\theta_1)(1-\cos 2\theta_2+2\theta_2\sin 2\theta_2)]$$

$$b_{24}=4\theta_1^2\cos 2\theta_1-\Gamma\frac{\kappa_1+1}{\kappa_2+1}\cdot 4\theta_2^2\cos 2\theta_2-2\frac{1-\Gamma}{\kappa_2+1}[4\theta_1^2\sin 2\theta_1\sin 2\theta_2$$
$$+(1-\cos 2\theta_1-2\theta_1\sin 2\theta_1)(1-\cos 2\theta_2-2\theta_2\sin 2\theta_2)$$
$$-(-\sin 2\theta_1+2\theta_1\cos 2\theta_1)(\sin 2\theta_2+2\theta_2\cos 2\theta_2)] \tag{26}$$

and the following relations exist: $b_{11}\cdot b_{22}-b_{21}\cdot b_{12}=\Theta_1$, $b_{11}\sin 2\theta_2-b_{21}(1-\cos 2\theta_2)=\Delta$ and $b_{12}\sin 2\theta_2-b_{22}(1-\cos 2\theta_2)=Q_1\sin 2\theta_2-Q_2(1)-\cos 2\theta_2)$.

Noticing $\Theta_1=b_{11}b_{22}-b_{21}b_{12}=0$ and making use of eqns (24), the constant E_1 can be obtained through subtracting (25b) multiplied by b_{11} from (25a) multiplied by b_{21}, and the constant $C_1+\overline{C_1}/2$ through subtracting (25b) multiplied by b_{12} from (25a) multiplied by b_{22}, the results are

$$E_1=i\frac{1}{\Theta_2}\cdot\frac{4\mu_2}{\kappa_2+1}(\alpha_2^*-\alpha_1^*)\,\Delta T\cdot\Delta$$
$$\frac{C_1+\overline{C_1}}{2}=\frac{1}{\Theta_2}\cdot\frac{4\mu_2}{\kappa_2+1}(\alpha_2^*-\alpha_1^*)\Delta T\cdot[Q_1\sin 2\theta_2-Q_2(1-\cos 2\theta_2)] \tag{27}$$

where

$$\Theta_2 = b_{22}b_{13} - b_{12}b_{23} - b_{21}b_{14} + b_{11}b_{24} \tag{28}$$

Substituting (27) into (25) yields

$$b_{11}\left\{(A_1 - \overline{A_1}) - \frac{1}{\Theta_2}\cdot\frac{4\mu_2}{\kappa_2+1}(\alpha_2^* - \alpha_1^*)\,\Delta T\cdot[b_{14}\sin 2\theta_2 - b_{24}(1-\cos 2\theta_2)]\right\}$$
$$+ b_{12}\left\{\frac{C_1 - \overline{C_1}}{2}i + \frac{1}{\Theta_2}\cdot\frac{4\mu_2}{\kappa_2+1}(\alpha_2^* - \alpha_1^*)\Delta T\cdot[b_{13}\sin 2\theta_2 - b_{23}(1-\cos 2\theta_2)]\right\} = 0 \tag{29a}$$

$$b_{21}\left\{(A_1 + \overline{A_1}) - \frac{1}{\Theta_2}\cdot\frac{4\mu_2}{\kappa_2+1}(\alpha_2^* - \alpha_1^*)\Delta T\cdot[b_{14}\sin 2\theta_2 - b_{24}(1-\cos 2\theta_2)]\right\}$$
$$+ b_{22}\left\{\frac{C_1 - \overline{C_1}}{2}i + \frac{1}{\Theta_2}\cdot\frac{4\mu_2}{\kappa_2+1}(\alpha_2^* - \alpha_1^*)\Delta T\cdot[b_{13}\sin 2\theta_2 - b_{23}(1-\cos 2\theta_2)]\right\} = 0 \tag{29b}$$

Subtracting (29b) multiplied by $(1-\cos 2\theta_2)$ from (29a) multiplied by $\sin 2\theta_2$, one obtains

$$\Delta\cdot\left\{(A_1 + \overline{A_1}) - \frac{1}{\Theta_2}\cdot\frac{4\mu_2}{\kappa_2+1}(\alpha_2^* - \alpha_1^*)\Delta T\cdot[b_{14}\sin 2\theta_2 - b_{24}(1-\cos 2\theta_2)]\right\}$$
$$+[Q_1\sin 2\theta_2 - Q_2(1-\cos 2\theta_2)] \tag{30}$$
$$\cdot\left\{\frac{C_1 - \overline{C_1}}{2}i + \frac{1}{\Theta_2}\cdot\frac{4\mu_2}{\kappa_2+1}(\alpha_2^* - \alpha_1^*)\Delta T\cdot[b_{13}\sin 2\theta_2 - b_{23}(1-\cos 2\theta_2)]\right\} = 0$$

Because $\Theta_1 = b_{11}b_{22} - b_{21}b_{12} = 0$, only one is independent among the three equations (29a), (29b) and (30). Furthermore, noticing $\Delta \neq 0$, hence eqn (30) is independent of the other two, from it one obtains

$$\frac{C_1 - \overline{C_1}}{2} = ik_0\Delta + i\frac{1}{\Theta_2}\cdot\frac{4\mu_2}{\kappa_2+1}(\alpha_2^* - \alpha_1^*)\Delta T\cdot[b_{13}\sin 2\theta_2 - b_{23}(1-\cos 2\theta_2)]$$
$$A_1 + \overline{A_1} = k_0[Q_1\sin 2\theta_2 - Q_2(1-\cos 2\theta_2)]$$
$$+ \frac{1}{\Theta_2}\cdot\frac{4\mu_2}{\kappa_2+1}(\alpha_2^* - \alpha_1^*)\Delta T\cdot[b_{14}\sin 2\theta_2 - b_{24}(1-\cos 2\theta_2)] \tag{31}$$

where k_0 is an arbitrary real constant.

The complex constants E_2, C_2 and A_2 can be derived from (23), then from (11) $F_j = 0$ and D_j, B_j can be achieved. Finally, from (8) the solution for $\Theta_1 = 0$ is

$$\varphi_1(z) = \frac{A_1 - \overline{A_1}}{2}z + \frac{A_1 + \overline{A_1}}{2}z + C_1 z\ln z + E_1 z\ln^2 z$$
$$\psi_1(z) = B_1 z + D_1 z\ln z \tag{32a}$$

$$\varphi_2(z) = A_2 z + C_2 z\ln z + E_2 z\ln^2 z$$
$$\psi_2(z) = B_2 z + D_2 z\ln z \tag{32b}$$

where $A_1 - \overline{A_1}/2$ is an arbitrary imaginary constant, and the terms related to the arbitrary real constant k_0 represent a homogeneous solution. Equation (32) is just the solution for the paradox, which can be called the primary paradox solution.

Substituting (32) into (5) and noticing $E_j + \overline{E_j} = 0$ yields

$$\sigma_{j\theta} + i\tau_{jr\theta} = [1 - e^{2i(\theta-\theta_j)}] \cdot \{[(C_j + \overline{C_j}) + 2(1+2i\theta_j)E_j] \cdot (\ln r + i\theta + 1) + A_j + \overline{A_j} + C_j - 2i\theta_j\overline{C_j} + 4\theta_j^2 E_j\} + 4i(\theta-\theta_j)E_j(\ln r + 1 - i\theta_j) - 2i(\theta-\theta_j)\overline{C_j} \quad (j = 1,2) \tag{33}$$

It is observed that the singularity of $\ln r$ appears in thermal residual stresses.

However, analyses and computations demonstrate that the value of Θ_2 still possibly equals zero when $\Theta_1 = 0$, and this may occur for $(\theta_1 - \theta_2) > 101.24°$. For instance, if $\theta_1 = 105°$, $\theta_2 = -90°$, $\alpha = 0.8734$, $\beta = 0.4314$, then $\Theta_1 = 0$ and $\Theta_2 = 0$, thereby the primary paradox solution (32) is still infinite, this is the secondary paradox.

3.3. *The secondary paradox solution (when $\Theta_1 = \Theta_2 = 0$)*

Taking $M_j = N_j = 0$, $(j = 1,2)$. From equations (12b) and (13) we have $G_j + \overline{G_j} = 0$ and

$$G_2 = \Gamma\frac{\kappa_1+1}{\kappa_2+1}G_1$$

$$E_2 = \Gamma\frac{\kappa_1+1}{\kappa_2+1}E_1 + \frac{1-\Gamma}{\kappa_2+1}[(E_1+\overline{E_1})(1-e^{2i\theta_1}) - 3G_1(1-e^{2i\theta_1}+1+2i\theta_1 e^{2i\theta_1})]$$

$$C_2 = \Gamma\frac{\kappa_1+1}{\kappa_2+1}C_1 + \frac{1-\Gamma}{\kappa_2+1}[(C_1+\overline{C_1})(1-e^{2i\theta_1}) - (E_1-\overline{E_1})(1-e^{2i\theta_1}+2i\theta_1 e^{2i\theta_1}) + (E_1+\overline{E_1})(1-e^{2i\theta_1}-2i\theta_1 e^{2i\theta_1}) + G_1 \cdot 12\theta_1^2 e^{2i\theta_1}]$$

$$A_2 = \Gamma\frac{\kappa_1+1}{\kappa_2+1}A_1 + \frac{1-\Gamma}{\kappa_2+1}\left[(A_1+\overline{A_1})(1-e^{2i\theta_1}) - \frac{C_1-\overline{C_1}}{2}(1-e^{2i\theta_1}+2i\theta_1 e^{2i\theta_1}) + \frac{C_1+\overline{C_1}}{2}(1-e^{2i\theta_1}-2i\theta_1 e^{2i\theta_1}) + E_1 \cdot 4\theta_1^2 e^{2i\theta_1} + G_1 \cdot 8i\theta_1^3 e^{2i\theta_1}\right] - \frac{2\mu_2}{\kappa_2+1}(\alpha_2^* - \alpha_1^*)\Delta T \tag{34}$$

Substituting (34) into (12c), (12d) and (12e), then separating the real parts from the imaginary parts, we obtain

$$b_{11} \cdot (E_1+\overline{E_1}) + b_{12} \cdot 3G_1 i = 0 \tag{35a}$$

$$b_{21} \cdot (E_1+\overline{E_1}) + b_{22} \cdot 3G_1 i = 0 \tag{35b}$$

$$b_{11} \cdot (C_1+\overline{C_1}) + b_{12} \cdot (E_1-\overline{E_1})i + b_{13} \cdot (E_1+\overline{E_1}) + b_{14} \cdot 3G_1 i = 0 \tag{36a}$$

$$b_{21} \cdot (C_1+\overline{C_1}) + b_{22} \cdot (E_1-\overline{E_1})i + b_{23} \cdot (E_1+\overline{E_1}) + b_{24} \cdot 3G_1 i = 0 \tag{36b}$$

$$b_{11}\cdot(A_1+\overline{A_1})+b_{12}\cdot\frac{C_1-\overline{C_1}}{2}i+b_{13}\cdot\frac{C_1+\overline{C_1}}{2}+b_{14}\cdot\frac{E_1-\overline{E_1}}{2}i$$
$$+b_{15}\cdot\frac{E_1+\overline{E_1}}{2}+b_{16}\cdot\frac{3}{2}G_1 i=-\frac{4\mu_2}{\kappa_2+1}(\alpha_2^*-\alpha_1^*)\Delta T(1-\cos 2\theta_2) \quad (37a)$$

$$b_{21}\cdot(A_1+\overline{A_1})+b_{22}\cdot\frac{C_1-\overline{C_1}}{2}i+b_{23}\cdot\frac{C_1+\overline{C_1}}{2}+b_{24}\cdot\frac{E_1-\overline{E_1}}{2}i$$
$$+b_{25}\cdot\frac{E_1+\overline{E_1}}{2}+b_{26}\cdot\frac{3}{2}G_1 i=-\frac{4\mu_2}{\kappa_2+1}(\alpha_2^*-\alpha_1^*)\Delta T\sin 2\theta_2 \quad (37b)$$

where

$$b_{15}=4\theta_1^2\cos 2\theta_1-4\theta_2^2\cos 2\theta_2\left[\Gamma\frac{\kappa_1+1}{\kappa_2+1}+2\frac{1-\Gamma}{\kappa_2+1}(1-\cos 2\theta_1)\right]-2\frac{1-\Gamma}{\kappa_2+1}$$
$$\cdot[4\theta_1^2\cos 2\theta_1(1-\cos 2\theta_2)-(\sin 2\theta_1+2\theta_1\cos 2\theta_1)(-\sin 2\theta_2+2\theta_2\cos 2\theta_2)$$
$$+(1-\cos 2\theta_1+2\theta_1\sin 2\theta_1)(1-\cos 2\theta_2+2\theta_2\sin 2\theta_2)+4\theta_2^2\sin 2\theta_1\sin 2\theta_2]$$
$$b_{25}=-4\theta_1^2\sin 2\theta_1+4\theta_2^2\sin 2\theta_2\left[\Gamma\frac{\kappa_1+1}{\kappa_2+1}+2\frac{1-\Gamma}{\kappa_2+1}(1-\cos 2\theta_1)\right]-2\frac{1-\Gamma}{\kappa_2+1}$$
$$\cdot[4\theta_1^2\cos 2\theta_1\sin 2\theta_2-(\sin 2\theta_1+2\theta_1\cos 2\theta_1)(1-\cos 2\theta_2-2\theta_2\sin 2\theta_2)$$
$$+(1-\cos 2\theta_1+2\theta_1\sin 2\theta_1)(\sin 2\theta_2+2\theta_2\cos 2\theta_2)+4\theta_2^2\sin 2\theta_1\cos 2\theta_2] \quad (38a)$$

$$b_{16}=\frac{2}{3}\left\{8\theta_1^3\cos 2\theta_1-\Gamma\frac{\kappa_1+1}{\kappa_2+1}\cdot 8\theta_2^3\cos 2\theta_2-\frac{1-\Gamma}{\kappa_2+1}[16\theta_1^3\cos 2\theta_1(1-\cos 2\theta_2)\right.$$
$$-12\theta_1^2\cos 2\theta_1(-\sin 2\theta_2+2\theta_2\cos 2\theta_2)+12\theta_1^2\sin 2\theta_1(1-\cos 2\theta_2+2\theta_2\sin 2\theta_2)$$
$$\left.-12(1-\cos 2\theta_1-2\theta_1\sin 2\theta_1)\theta_2^2\sin 2\theta_2-12(-\sin 2\theta_1+2\theta_1\cos 2\theta_1)\theta_2^2\cos 2\theta_2]\right\}$$

$$b_{26}=\frac{2}{3}\left\{-8\theta_1^3\sin 2\theta_1+\Gamma\frac{\kappa_1+1}{\kappa_2+1}\cdot 8\theta_2^3\sin 2\theta_2-\frac{1-\Gamma}{\kappa_2+1}[16\theta_1^3\cos 2\theta_1\sin 2\theta_2\right.$$
$$-12\theta_1^2\cos 2\theta_1(1-\cos 2\theta_2-2\theta_2\sin 2\theta_2)+12\theta_1^2\sin 2\theta_1(\sin 2\theta_2+2\theta_2\cos 2\theta_2)$$
$$\left.-12(1-\cos 2\theta_1-2\theta_1\sin 2\theta_1)\theta_2^2\cos 2\theta_2+12(-\sin 2\theta_1+2\theta_1\cos 2\theta_1)\theta_2^2\sin 2\theta_2]\right\} \quad (38b)$$

Applying the method similar to that in 3.2 to solve the set of eqns (35)–(37), and noticing $\Theta_1=\Theta_2=0$, we get

$$G_1=i\frac{2}{3\Theta_3}\cdot\frac{4\mu_2}{\kappa_2+1}(\alpha_2^*-\alpha_1^*)\Delta T\cdot\Delta$$

$$E_1-\overline{E_1} = \frac{2}{\Theta_3}\cdot\frac{4\mu_2}{\kappa_2+1}(\alpha_2^*-\alpha_1^*)\Delta T\cdot[Q_1\sin 2\theta_2-Q_2(1-\cos 2\theta_2)]$$

$$E_1-\overline{E_1} = i\cdot 2k\Delta+i\frac{2}{\Theta_3}\cdot\frac{4\mu_2}{\kappa_2+1}(\alpha_2^*-\alpha_1^*)\Delta T\cdot[b_{13}\sin 2\theta_2-b_{23}(1-\cos 2\theta_2)]$$

$$C_1+\overline{C_1} = 2k[Q_1\sin 2\theta_2-Q_2(1-\cos 2\theta_2)] + \frac{2}{\Theta_3}\cdot\frac{4\mu_2}{\kappa_2+1}(\alpha_2^*-\alpha_1^*)\Delta T\cdot[b_{14}\sin 2\theta_2-b_{24}(1-\cos 2\theta_2)]$$

$$\frac{C_1-\overline{C_1}}{2} = ik_0\Delta+ik[b_{13}\sin 2\theta_2-b_{23}(1-\cos 2\theta_2)] + i\frac{1}{\Theta_3}\cdot\frac{4\mu_2}{\kappa_2+1}(\alpha_2^*-\alpha_1^*)\Delta T\cdot[b_{15}\sin 2\theta_2-b_{25}(1-\cos 2\theta_2)]$$

$$A_1+\overline{A_1} = k_0[Q_1\sin 2\theta_2-Q_2(1-\cos 2\theta_2)]+k[b_{14}\sin 2\theta_2-b_{24}(1-\cos 2\theta_2)] + \frac{1}{\Theta_3}\cdot\frac{4\mu_2}{\kappa_2+1}(\alpha_2^*-\alpha_1^*)\Delta T\cdot[b_{16}\sin 2\theta_2-b_{26}(1-\cos 2\theta_2)] \quad (39)$$

where

$$\Theta_3 = b_{11}b_{26}-b_{21}b_{16}-b_{12}b_{25}+b_{22}b_{15}+b_{13}b_{24}-b_{23}b_{14} \quad (40)$$

k and k_0 are two independent arbitrary real constants.

The complex constants G_2, E_2, C_2 and A_2 can be derived from (34), then from (11) $H_j = 0$ and F_j, D_j, B_j can be achieved. Finally, from (8) the solution for $\Theta_1 = \Theta_2 = 0$ is

$$\varphi_1(z) = \frac{A_1-\overline{A_1}}{2}z+\frac{A_1+\overline{A_1}}{2}z+C_1 z\ln z+E_1 z\ln^2 z+G_1 z\ln^3 z$$

$$\psi_1(z) = B_1 z+D_1 z\ln z+F_1 z\ln^2 z \quad (41a)$$

$$\varphi_2(z) = A_2 z+C_2 z\ln z+E_2 z\ln^2 z+G_2 z\ln^3 z$$

$$\psi_2(z) = B_2 z+D_2 z\ln z+F_2 z\ln^2 z \quad (41b)$$

where $A_1-\overline{A_1}/2$ is an arbitrary imaginary constant, the terms related to the arbitrary real constant k or k_0 represent a homogeneous solution. Equation (41) is just the solution for the secondary paradox, which can be called the secondary paradox solution.

Substituting (41) into (5) and noticing $G_j+\overline{G_j} = 0$ yields

$$\sigma_{j\theta}+i\tau_{jr\theta} = [1-e^{2i(\theta-\theta_j)}]\{(E_j+\overline{E_j}+3G_j+6i\theta_j G_j)[\ln^2 r+2i\theta\ln r-\theta^2+2(\ln r+i\theta)] + (C_j+\overline{C_j}+2E_j-4i\theta_j\overline{E_j}+12\theta_j^2 G_j)\cdot(\ln r+i\theta+1)+(A_j+\overline{A_j}+C_j - 2i\theta_j\overline{C_j}-4\theta_j^2\overline{E_j}-8i\theta_j^3 G_j)\}+2i(\theta-\theta_j)G_j[3\ln^2 r+6(1-i\theta_j)\ln r$$

$$-(\theta^2-2\theta\theta_j+4\theta_j^2+6i\theta_j)]-4i(\theta-\theta_j)\overline{E_j}(\ln r+1-i\theta_j)-2i(\theta-\theta_j)\overline{C_j}\quad (j=1,2) \tag{42}$$

it is observed that the singularities of $\ln^2 r$ and $\ln r$ appear in thermal residual stresses.

However, numerical analyses demonstrate that the value of Θ_3 still possibly equals zero when $\Theta_1=0$ and $\Theta_2=0$. For instance, if $\theta_1=182.7099°$, $\theta_2=-100°$, $\alpha=-0.9989$, $\beta=-0.4853$, then $\Theta_1=\Theta_2=0$ and $\Theta_3=0$, thereby the secondary paradox solution (41) is still infinite, this is the triple paradox.

3.4. The triple paradox solution (when $\Theta_1=\Theta_2=\Theta_3=0$)

From eqns (12a) and (13) we have $M_j+\overline{M_j}=0$ and

$$M_2=\Gamma\frac{\kappa_1+1}{\kappa_2+1}M_1$$

$$G_2=\Gamma\frac{\kappa_1+1}{\kappa_2+1}G_1+\frac{1-\Gamma}{\kappa_2+1}[(G_1+\overline{G_1})(1-e^{2i\theta_1})-4M_1(1-e^{2i\theta_1}+2i\theta_1e^{2i\theta_1})]$$

$$E_2=\Gamma\frac{\kappa_1+1}{\kappa_2+1}E_1+\frac{1-\Gamma}{\kappa_2+1}\left[(E_1+\overline{E_1})(1-e^{2i\theta_1})-3\cdot\frac{G_1-\overline{G_1}}{2}(1-e^{2i\theta_1}+2i\theta_1e^{2i\theta_1})\right.$$
$$\left.+3\cdot\frac{G_1+\overline{G_1}}{2}(1-e^{2i\theta_1}-2i\theta_1e^{2i\theta_1})+M_1\cdot24\theta_1^2e^{2i\theta_1}\right]$$

$$C_2=\Gamma\frac{\kappa_1+1}{\kappa_2+1}C_1+\frac{1-\Gamma}{\kappa_2+1}[(C_1+\overline{C_1})(1-e^{2i\theta_1})-(E_1-\overline{E_1})(1-e^{2i\theta_1}+2i\theta_1e^{2i\theta_1})$$
$$+(E_1+\overline{E_1})(1-e^{2i\theta_1}-2i\theta_1e^{2i\theta_1})+G_1\cdot12\theta_1^2e^{2i\theta_1}+M_1\cdot32i\theta_1^3e^{2i\theta_1}]$$

$$A_2=\Gamma\frac{\kappa_1+1}{\kappa_2+1}A_1+\frac{1-\Gamma}{\kappa_2+1}\left[(A_1+\overline{A_1})(1-e^{2i\theta_1})-\frac{C_1-\overline{C_1}}{2}(1-e^{2i\theta_1}+2i\theta_1e^{2i\theta_1})\right.$$
$$\left.+\frac{C_1+\overline{C_1}}{2}(1-e^{2i\theta_1}-2i\theta_1e^{2i\theta_1})+E_1\cdot4\theta_1^2e^{2i\theta_1}+G_1\cdot8i\theta_1^3e^{2i\theta_1}-M_1\cdot16\theta_1^4e^{2i\theta_1}\right]$$
$$-\frac{2\mu_2}{\kappa_2+1}(\alpha_2^*-\alpha_1^*)\Delta T \tag{43}$$

Introducing (43) in (12b)–(12e), then separating the real parts from the imaginary parts, we obtain

$$b_{11}\cdot(G_1+\overline{G_1})+b_{12}\cdot4M_1i=0 \tag{44a}$$

$$b_{21}\cdot(G_1+\overline{G_1})+b_{22}\cdot4M_1i=0 \tag{44b}$$

$$b_{11}\cdot(E_1+\overline{E_1})+b_{12}\cdot\frac{3}{2}(G_1-\overline{G_1})i+b_{13}\cdot\frac{3}{2}(G_1+\overline{G_1})+b_{14}\cdot6M_1i=0 \tag{45a}$$

$$b_{21}\cdot(E_1+\overline{E_1})+b_{22}\cdot\frac{3}{2}(G_1-\overline{G_1})i+b_{23}\cdot\frac{3}{2}(G_1+\overline{G_1})+b_{24}\cdot 6M_1 i=0 \tag{45b}$$

$$b_{11}\cdot(C_1+\overline{C_1})+b_{12}\cdot(E_1-\overline{E_1})i+b_{13}\cdot(E_1+\overline{E_1})+b_{14}\cdot\frac{3}{2}(G_1-\overline{G_1})i + b_{15}\cdot\frac{3}{2}(G_1+\overline{G_1})+b_{16}\cdot 6M_1 i=0 \tag{46a}$$

$$b_{21}\cdot(C_1+\overline{C_1})+b_{22}\cdot(E_1-\overline{E_1})i+b_{23}\cdot(E_1+\overline{E_1})+b_{24}\cdot\frac{3}{2}(G_1-\overline{G_1})i + b_{25}\cdot\frac{3}{2}(G_1+\overline{G_1})+b_{26}\cdot 6M_1 i=0 \tag{46b}$$

$$\begin{aligned}&b_{11}\cdot(A_1+\overline{A_1})+b_{12}\cdot\frac{C_1-\overline{C_1}}{2}i+b_{13}\cdot\frac{C_1+\overline{C_1}}{2}+b_{14}\cdot\frac{E_1-\overline{E_1}}{2}i\\ &+b_{15}\cdot\frac{E_1+\overline{E_1}}{2}+b_{16}\cdot\frac{3}{4}(G_1-\overline{G_1})i+b_{17}\cdot\frac{3}{4}(G_1+\overline{G_1})+b_{18}\cdot 3M_1 i\\ &=-\frac{4\mu_2}{\kappa_2+1}(\alpha_2^*-\alpha_1^*)\Delta T(1-\cos 2\theta_2)\end{aligned} \tag{47a}$$

$$\begin{aligned}&b_{21}\cdot(A_1+\overline{A_1})+b_{22}\cdot\frac{C_1-\overline{C_1}}{2}i+b_{23}\cdot\frac{C_1+\overline{C_1}}{2}+b_{24}\cdot\frac{E_1-\overline{E_1}}{2}i\\ &+b_{25}\cdot\frac{E_1+\overline{E_1}}{2}+b_{26}\cdot\frac{3}{4}(G_1-\overline{G_1})i+b_{27}\cdot\frac{3}{4}(G_1+\overline{G_1})+b_{28}\cdot 3M_1 i\\ &=-\frac{4\mu_2}{\kappa_2+1}(\alpha_2^*-\alpha_1^*)\Delta T\sin 2\theta_2\end{aligned} \tag{47b}$$

where

$$\begin{aligned}b_{17}=\frac{2}{3}\Bigg\{&-8\theta_1^3\sin 2\theta_1+\left[\Gamma\frac{\kappa_1+1}{\kappa_2+1}+2\frac{1-\Gamma}{\kappa_2+1}(1-\cos 2\theta_1)\right]\cdot 8\theta_2^3\sin 2\theta_2\\ &+\frac{1-\Gamma}{\kappa_2+1}[16\theta_1^3\sin 2\theta_1\cdot(1-\cos 2\theta_2)-12\theta_1^2\sin 2\theta_1\cdot(-\sin 2\theta_2+2\theta_2\cos 2\theta_2)\\ &-12\theta_1^2\cos 2\theta_1\cdot(1-\cos 2\theta_2+2\theta_2\sin 2\theta_2)-12(\sin 2\theta_1+2\theta_1\cos 2\theta_1)\theta_2^2\sin 2\theta_2\\ &-12(1-\cos 2\theta_1+2\theta_1\sin 2\theta_1)\theta_2^2\cos 2\theta_2-16\sin 2\theta_1\cdot\theta_2^3\cos 2\theta_2]\Bigg\}\end{aligned}$$

$$b_{27}=\frac{2}{3}\Bigg\{-8\theta_1^3\cos 2\theta_1+\left[\Gamma\frac{\kappa_1+1}{\kappa_2+1}+2\frac{1-\Gamma}{\kappa_2+1}(1-\cos 2\theta_1)\right]\cdot 8\theta_2^3\cos 2\theta_2$$

$$+\frac{1-\Gamma}{\kappa_2+1}[16\theta_1^3\sin 2\theta_1\cdot\sin 2\theta_2-12\theta_1^2\sin 2\theta_1\cdot(1-\cos 2\theta_2-2\theta_2\sin 2\theta_2)$$
$$-12\theta_1^2\cos 2\theta_1\cdot(\sin 2\theta_2+2\theta_2\cos 2\theta_2)-12(\sin 2\theta_1+2\theta_1\cos 2\theta_1)\theta_2^2\cos 2\theta_2$$
$$\left.+12(1-\cos 2\theta_1+2\theta_1\sin 2\theta_1)\theta_2^2\sin 2\theta_2+16\sin 2\theta_1\cdot\theta_2^3\sin 2\theta_2]\right\} \quad (48a)$$

$$b_{18}=\frac{2}{3}\left\{-8\theta_1^4\sin 2\theta_1+\Gamma\frac{\kappa_1+1}{\kappa_2+1}\cdot 8\theta_2^4\sin 2\theta_2+\frac{1-\Gamma}{\kappa_2+1}[16\theta_1^4\sin 2\theta_1\cdot(1-\cos 2\theta_2)\right.$$
$$-16\theta_1^3\sin 2\theta_1(-\sin 2\theta_2+2\theta_2\cos 2\theta_2)-16\theta_1^3\cos 2\theta_1(1-\cos 2\theta_2+2\theta_2\sin 2\theta_2)$$
$$-48\theta_1^2\theta_2^2\cos 2\theta_1\cdot\sin 2\theta_2-48\theta_1^2\theta_2^2\sin 2\theta_1\cdot\cos 2\theta_2$$
$$\left.+16(1-\cos 2\theta_1-2\theta_1\sin 2\theta_1)\theta_2^3\cos 2\theta_2-16(-\sin 2\theta_1+2\theta_1\cos 2\theta_1)\theta_2^3\sin 2\theta_2]\right\}$$

$$b_{28}=\frac{2}{3}\left\{-8\theta_1^4\cos 2\theta_1+\Gamma\frac{\kappa_1+1}{\kappa_2+1}\cdot 8\theta_2^4\cos 2\theta_2+\frac{1-\Gamma}{\kappa_2+1}[16\theta_1^4\sin 2\theta_1\cdot\sin 2\theta_2\right.$$
$$-16\theta_1^3\sin 2\theta_1(1-\cos 2\theta_2-2\theta_2\sin 2\theta_2)-16\theta_1^3\cos 2\theta_1(\sin 2\theta_2+2\theta_2\cos 2\theta_2)$$
$$-48\theta_1^2\theta_2^2\cos 2\theta_1\cdot\cos 2\theta_2+48\theta_1^2\theta_2^2\sin 2\theta_1\cdot\sin 2\theta_2$$
$$\left.-16(1-\cos 2\theta_1-2\theta_1\sin 2\theta_1)\theta_2^3\sin 2\theta_2-16(-\sin 2\theta_1+2\theta_1\cos 2\theta_1)\theta_2^3\cos 2\theta_2]\right\} \quad (48b)$$

Applying the method similar to that in 3.2 to solve the set of eqns (44)–(47), and noticing $\Theta_1=\Theta_2=\Theta_3=0$, we get

$$M_1=i\frac{1}{3\Theta_4}\cdot\frac{4\mu_2}{\kappa_2+1}(\alpha_2^*-\alpha_1^*)\Delta T\cdot\Delta$$
$$\frac{G_1+\overline{G_1}}{2}=\frac{2}{3\Theta_4}\cdot\frac{4\mu_2}{\kappa_2+1}(\alpha_2^*-\alpha_1^*)\Delta T\cdot[Q_1\sin 2\theta_2-Q_2(1-\cos 2\theta_2)]$$
$$\frac{G_1-\overline{G_1}}{2}=i\cdot\frac{2}{3}k^*\Delta+i\frac{2}{3\Theta_4}\cdot\frac{4\mu_2}{\kappa_2+1}(\alpha_2^*-\alpha_1^*)\Delta T\cdot[b_{13}\sin 2\theta_2-b_{23}(1-\cos 2\theta_2)]$$
$$E_1+\overline{E_1}=2k^*[Q_1\sin 2\theta_2-Q_2(1-\cos 2\theta_2)]$$
$$+\frac{2}{\Theta_4}\cdot\frac{4\mu_2}{\kappa_2+1}(\alpha_2^*-\alpha_1^*)\Delta T\cdot[b_{14}\sin 2\theta_2-b_{24}(1-\cos 2\theta_2)]$$
$$E_1-\overline{E_1}=i\cdot 2k\Delta+i\cdot 2k^*[b_{13}\sin 2\theta_2-b_{23}(1-\cos 2\theta_2)]$$
$$+i\frac{2}{\Theta_4}\cdot\frac{4\mu_2}{\kappa_2+1}(\alpha_2^*-\alpha_1^*)\Delta T\cdot[b_{15}\sin 2\theta_2-b_{25}(1-\cos 2\theta_2)]$$

$$C_1+\overline{C_1} = 2k[Q_1\sin 2\theta_2-Q_2(1-\cos 2\theta_2)]+2k^*[b_{14}\sin 2\theta_2+b_{24}(1-\cos 2\theta_2)]$$
$$+\frac{2}{\Theta_4}\cdot\frac{4\mu_2}{\kappa_2+1}(\alpha_2^*-\alpha_1^*)\Delta T\cdot[b_{16}\sin 2\theta_2-b_{26}(1-\cos 2\theta_2)]$$

$$\frac{C_1-\overline{C_1}}{2} = ik_0\Delta$$
$$+ik[b_{13}\sin 2\theta_2-b_{23}(1-\cos 2\theta_2)]+ik^*[b_{15}\sin 2\theta_2-b_{25}(1-\cos 2\theta_2)]$$
$$+i\frac{1}{\Theta_4}\cdot\frac{4\mu_2}{\kappa_2+1}(\alpha_2^*-\alpha_1^*)\Delta T\cdot[b_{17}\sin 2\theta_2-b_{27}(1-\cos 2\theta_2)]$$

$$A_1+\overline{A_1} = k_0[Q_1\sin 2\theta_2-Q_2(1-\cos 2\theta_2)]$$
$$+k[b_{14}\sin 2\theta_2-b_{24}(1-\cos 2\theta_2)]+k^*[b_{16}\sin 2\theta_2-b_{26}(1-\cos 2\theta_2)]$$
$$+\frac{1}{\Theta_4}\cdot\frac{4\mu_2}{\kappa_2+1}(\alpha_2^*-\alpha_1^*)\Delta T\cdot[b_{18}\sin 2\theta_2-b_{28}(1-\cos 2\theta_2)] \tag{49}$$

where

$$\Theta_4 = b_{11}b_{28}-b_{21}b_{18}-b_{12}b_{27}+b_{22}b_{17}+b_{13}b_{26}-b_{23}b_{16}-b_{14}b_{25}+b_{24}b_{15} \tag{50}$$

k^*, k and k_0 are three independent, arbitrary real constants.

The complex constants M_2, G_2, E_2, C_2 and A_2 can be derived from (43) then from (11) $N_j = 0$ and H_j, F_j, D_j, B_j can be achieved. Finally, from (8) the solution for $\Theta_1 = \Theta_2 = \Theta_3 = 0$ is

$$\varphi_1(z) = \frac{A_1-\overline{A_1}}{2}z+\frac{A_1+\overline{A_1}}{2}z+C_1z\ln z+E_1z\ln^2 z+G_1z\ln^3 z+M_1z\ln^4 z$$
$$\psi_1(z) = B_1z+D_1z\ln z+F_1z\ln^2 z+H_1z\ln^2 z \tag{51a}$$

$$\varphi_2(z) = A_2z+C_2z\ln z+E_2z\ln^2 z+G_2z\ln^3 z+M_2z\ln^4 z$$
$$\psi_2(z) = B_2z+D_2z\ln z+F_2z\ln^2 z+H_2z\ln^3 z \tag{51b}$$

where $A_1-\overline{A_1}/2$ is an arbitrary imaginary constant, the terms related to the arbitrary real constant k^*, k and k_0 represent a homogeneous solution. Equation (51) is just the solution for the triple paradox, which can be called the triple paradox solution.

Substituting (51) into (5) and noticing $M_j+\overline{M_j} = 0$ yields

$$\sigma_{j\theta}+i\tau_{jr\theta} = [1-e^{2i(\theta-\theta_j)}]\{(G_j+\overline{G_j}+4M_j+8i\theta_jM_j)[(\ln r+i\theta)^3+3(\ln r+i\theta)^2]$$
$$+(E_j+\overline{E_j}+3G_j-6i\theta_j\overline{G_j}+24\theta_j^2M_j)[(\ln r+i\theta)^2+2(\ln r+i\theta)]$$
$$+(C_j+\overline{C_j}+2E_j-4i\theta_j\overline{E_j}-12\theta_j^2\overline{G_j}-32i\theta_j^3M_j)\cdot(\ln r+i\theta+1)$$
$$+(A_j+\overline{A_j}+C_j-2i\theta_j\overline{C_j}-4\theta_j^2\overline{E_j}+8i\theta_j^3\overline{G_j}-16\theta_j^4M_j)\}$$
$$+8i(\theta-\theta_j)M_j[\ln^3 r+3(1-i\theta_j)\ln^2 r-(\theta^2-2\theta\theta_j+4\theta_j^2+6i\theta_j)\ln r$$

$$-(\theta^2-2\theta\theta_j+4\theta_j^2)+i\theta_j(\theta^2-2\theta\theta_j+2\theta_j^2)]-2i(\theta-\theta_j)\overline{G_j}[3\ln^2 r$$
$$+6(1-i\theta_j)\ln r-(\theta^2-2\theta\theta_j+4\theta_j^2+6i\theta_j)]-4i(\theta-\theta_j)\overline{E_j}(\ln r$$
$$+1-i\theta_j)-2i(\theta-\theta_j)\overline{C_j}\quad (j=1,2) \tag{52}$$

it is observed that the singularities of $\ln^3 r$, $\ln^2 r$ and $\ln r$ appear in thermal residual stresses.

Numerical analysis demonstrates that the triple paradox probably occurs only when the value of $(\theta_1-\theta_2)$ is in the range of 281.24°–284.34° or 355.55°–358.69° (for plain stress), and Θ_4, the denominator of the solution (51), does not vanish when $\Theta_1=\Theta_2=\Theta_3=0$, hence the paradox does not exist for the triple paradox solution (51).

Up till now, all the cases of $\Delta \neq 0$ have been studied and the corresponding particular solutions presented.

4. The particular solutions when $\Delta = 0$

Noticing that if two of the three terms $\sin\theta_1$, $\sin\theta_2$ and $\sin(\theta_1-\theta_2)$ equal zero, the third one definitely vanishes, there exist the following different circumstances:

4.1. $\sin(\theta_1-\theta_2)=0$ and $\sin\theta_1\neq 0$, $\sin\theta_2\neq 0$

In this case $\theta_1-\theta_2=\pi$ or 2π, $\theta_1\neq\pi$, $\theta_2\neq-\pi$, hence $1-e^{2i\theta_1}=1-e^{2i\theta_2}\neq 0$.

4.1.1. *The classical solution*

Taking $C_j=D_j=E_j=F_j=G_j=H_j=M_j=N_j=0$, $(j=1,2)$. Solving eqns (12) and (13), and from eqns(11) one obtains

$$\frac{A_1+\overline{A_1}}{2}=\frac{1}{\Delta_1}\cdot\frac{2\mu_2}{\kappa_2+1}(\alpha_2^*-\alpha_1^*)\Delta T$$
$$A_2=\Gamma\frac{\kappa_1+1}{\kappa_2+1}\cdot\frac{A_1-\overline{A_1}}{2}+\frac{1}{\Delta_1}\cdot\frac{2\mu_2}{\kappa_2+1}(\alpha_2^*-\alpha_1^*)\Delta T\left(1-i\cdot 2\frac{1-\Gamma}{\kappa_2+1}\sin 2\theta_1\right)$$
$$B_1=-\frac{1}{\Delta_1}\cdot\frac{4\mu_2}{\kappa_2+1}(\alpha_2^*-\alpha_1^*)\Delta T\cdot e^{-2i\theta_1}$$
$$B_2=-\frac{1}{\Delta_1}\cdot\frac{4\mu_2}{\kappa_2+1}(\alpha_2^*-\alpha_1^*)\Delta T\cdot e^{-2i\theta_2} \tag{53}$$

where

$$\Delta_1=\Gamma\frac{\kappa_1+1}{\kappa_2+1}+2\frac{1-\Gamma}{\kappa_2+1}(1-\cos 2\theta_1)-1=\frac{2}{1-\alpha}[\beta-(\beta-\alpha)\cos 2\theta_1] \tag{54}$$

thus from (8) we have

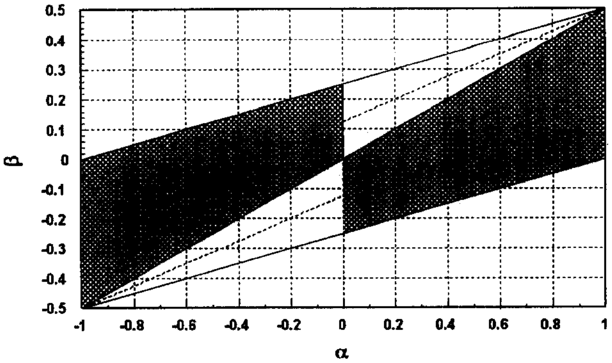

Fig. 3. The region in which the point (α, β) lies when $\Delta_1 = 0$.

$$\left.\begin{aligned}\varphi_1(z) &= \frac{A_1-\overline{A_1}}{2}z+\frac{A_1+\overline{A_1}}{2}z\\ \psi_1(z) &= B_1 z\end{aligned}\right\} \quad \text{and} \quad \left.\begin{aligned}\varphi_2(z) &= A_2 z\\ \psi_2(z) &= B_2 z\end{aligned}\right\} \tag{55}$$

where $A_1-\overline{A_1}/2$ is an arbitrary imaginary constant, (55) is just the classical solution.

Substituting (55) into (4) and (5) yields

$$\left.\begin{aligned}\sigma_{jr}-i\tau_{jr\theta} &= (A_j+\overline{A_j})[1+e^{2i(\theta-\theta_j)}]\\ \sigma_{j\theta}+i\tau_{jr\theta} &= (A_j+\overline{A_j})[1-e^{2i(\theta-\theta_j)}]\end{aligned}\right\} \quad (j=1,2) \tag{56}$$

Equations (55) and (56) are valid only when $\Delta_1 \neq 0$, on this condition the expression (56), which is a particular solution of thermal residual stresses, does not contain $\ln r$ terms, hence the logarithmic singularity will not appear.

When $\Delta_1 = 0$, i.e., $\cos 2\theta_1 = \beta/(\beta-\alpha)$, the classical solution (55) becomes infinite, this is a paradox, and the expression (56), the stresses in which are called thermoelastic constant stress terms by Ioka et al. (1996), breaks down. In this case the point (α, β) is in the shaded region shown in Fig. 3 (except the straight line $\alpha = 0$).

4.1.2. The primary paradox solution (*when* $\Delta_1 = 0$)

Taking $E_j = F_j = G_j = H_j = M_j = N_j = 0$, $(j = 1, 2)$. From (13) and using $\Delta_1 = 0$ we get

$$C_2\frac{C_1+\overline{C_1}}{2}+\Gamma\frac{\kappa_1+1}{\kappa_2+1}\cdot\frac{C_1-\overline{C_1}}{2}-i\frac{1-\Gamma}{\kappa_2+1}(C_1+\overline{C_1})\sin 2\theta_1 \tag{57a}$$

$$\begin{aligned}A_2 = {} & \frac{A_1+\overline{A_1}}{2}+\Gamma\frac{\kappa_1+1}{\kappa_2+1}\cdot\frac{A_1-\overline{A_1}}{2}-i\frac{1-\Gamma}{\kappa_2+1}(A_1+\overline{A_1})\sin 2\theta_1\\ & +\frac{1-\Gamma}{\kappa_2+1}\left[-\frac{C_1-\overline{C_1}}{2}\cdot(1-e^{2i\theta_1}+2i\theta_1 e^{2i\theta_1})+\frac{C_1+\overline{C_1})}{2}(1-e^{2i\theta_1}-2i\theta_1 e^{2i\theta_1})\right]\end{aligned}$$

$$-\frac{2\mu_2}{\kappa_2+1}(\alpha_2^*-\alpha_1^*)\Delta T \tag{57b}$$

it is easily observed that eqn (12d) is satisfied automatically.

Substituting (57) into (12e), then separating the real parts from the imaginary parts, one obtains

$$a_{11}\cdot\frac{C_1+\overline{C_1}}{2}+a_{12}\cdot\frac{C_1-\overline{C_1}}{2}i = -\frac{4\mu_2}{\kappa_2+1}(\alpha_2^*-\alpha_1^*)\Delta T(1-\cos 2\theta_2)$$

$$a_{21}\cdot\frac{C_1+\overline{C_1}}{2}+a_{22}\cdot\frac{C_1-\overline{C_1}}{2}i = -\frac{4\mu_2}{\kappa_2+1}(\alpha_2^*-\alpha_1^*)\Delta T\sin 2\theta_2 \tag{58}$$

where

$$a_{11} = -2\frac{1-\Gamma}{\kappa_2+1}(2-2\cos 2\theta_2+2\theta_2\sin 2\theta_2-4\theta_2\sin 2\theta_2\cos 2\theta_2)$$
$$+\Gamma\frac{\kappa_1+1}{\kappa_2+1}\cdot 2(\theta_1-\theta_2)\sin 2\theta_1$$

$$a_{21} = -2\frac{1-\Gamma}{\kappa_2+1}[2(\theta_1-\theta_2)(1-\cos 2\theta_1)+4\theta_2\sin^2 2\theta_1]+\Gamma\frac{\kappa_1+1}{\kappa_2+1}\cdot 2(\theta_1-\theta_2)\cos 2\theta_1$$

$$a_{12} = -\Gamma\frac{\kappa_1+1}{\kappa_2+1}\cdot 2(\theta_1-\theta_2)\cos 2\theta_1$$

$$a_{22} = -2\frac{1-\Gamma}{\kappa_2+1}(2-2\cos 2\theta_1-2\theta_1\sin 2\theta_1)+\Gamma\frac{\kappa_1+1}{\kappa_2+1}\cdot 2(\theta_1-\theta_2)\sin 2\theta_1 \tag{59}$$

Solving eqns (58) we have

$$\frac{C_1+\overline{C_1}}{2} = \frac{1}{\Delta_2}\cdot\frac{4\mu_2}{\kappa_2+1}(\alpha_2^*-\alpha_1^*)\Delta T\cdot[a_{12}\sin 2\theta_2-a_{22}(1-\cos 2\theta_2)]$$

$$\frac{C_1-\overline{C_1}}{2} = i\frac{1}{\Delta_2}\cdot\frac{4\mu_2}{\kappa_2+1}(\alpha_2^*-\alpha_1^*)\Delta T\cdot[a_{11}\sin 2\theta_2-a_{21}(1-\cos 2\theta_2)] \tag{60}$$

where

$$\Delta_2 = a_{11}a_{22}-a_{21}a_{12}$$
$$=\Gamma\frac{\kappa_1+1}{\kappa_2+1}\cdot 2(\theta_1-\theta_2)\left[2(\theta_1-\theta_2)+4\sin 2\theta_1\cdot\left(\Gamma\frac{\kappa_1+1}{\kappa_2+1}-1\right)\right]$$
$$+4\left(2\frac{1-\Gamma}{\kappa_2+1}\right)^2(1-\cos 2\theta_1-\theta_1\sin 2\theta_1)(1-\cos 2\theta_2+\theta_2\sin 2\theta_2-2\theta_2\sin 2\theta_2\cos 2\theta_2)$$
$$=\frac{4}{1-\alpha)^2}\{(1-\alpha^2)(\theta_1-\theta_2)^2+4\alpha(\theta_1-\theta_2)\sin 2\theta_1$$

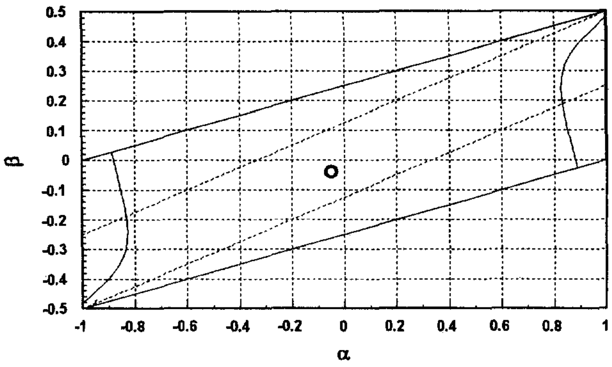

Fig. 4. The curve for $\theta_1-\theta_2=\pi$, on which the point (α,β) lies when $\Delta_1=\Delta_2=0$.

$$+4[\alpha^2+(\beta^2-\alpha^2)\theta_1\theta_2\sin^2 2\theta_1+\alpha\beta(\theta_1+\theta_2)\sin 2\theta_1]\} \tag{61}$$

Thus complex constant A_1 can be arbitrary, C_2 and A_2 can be derived from (57), then from (11) D_j and B_j can be achieved. Finally, from (8) the solution for $\Delta_1=0$ is

$$\left.\begin{aligned}\varphi_1(z)&=A_1z+C_1z\ln z\\ \psi_1(z)&=B_1z+D_1z\ln z\end{aligned}\right\}\quad\text{and}\quad\left.\begin{aligned}\varphi_2(z)&=A_2z+C_2z\ln z\\ \psi_2(z)&=B_2z+D_2z\ln z\end{aligned}\right\} \tag{62}$$

in which the terms related to the arbitrary complex constant A_1 represent a homogeneous solution, eqn (62) can be called the primary paradox solution.

Substituting (62) into (4) and (5) yields

$$\left.\begin{aligned}\sigma_{jr}-i\tau_{jr\theta}&=[1+e^{2i(\theta-\theta_j)}]\cdot[(C_j+\overline{C_j})(\ln r+i\theta+1)+A_j+\overline{A_j}\\&\quad+C_j-2i\theta_j\overline{C_j}]-2i(\theta-\theta_j)\overline{C_j}-2C_j\\ \sigma_{j\theta}+i\tau_{jr\theta}&=[1-e^{2i(\theta-\theta_j)}]\cdot[(C_j+\overline{C_j})(\ln r+i\theta+1)+A_j+\overline{A_j}\\&\quad+C_j-2i\theta_j\overline{C_j}]-2i(\theta-\theta_j)\overline{C_j}\end{aligned}\right\}\quad(j=1,2) \tag{63}$$

it is observed that the singularity of $\ln r$ appears in thermal residual stresses when $C_j+\overline{C_j}\neq 0$.

However, analyses and calculations demonstrate that the value of Δ_2 still possibly equals zero when $\Delta_1=0$. For instance, if $\theta_1=\pi/4$, $\theta_2=(3\pi)/(4)$, $\alpha=-(2\pi)/(4+\pi)$, $\beta=0$, then $\Delta_1=0$ and $\Delta_2=0$, thereby the primary paradox solution (62) is still infinite, this is the secondary paradox.

For $\theta_1-\theta_2=\pi$, the curve on which the point (α,β) lies when $\Delta_1=\Delta_2=0$ is shown in Fig. 4, and the two branches of it are polar symmetric about the origin O.

4.1.3. The secondary paradox solution (*when* $\Delta_1=\Delta_2=0$)

Taking $G_j=H_j=M_j=N_j=0$, $(j=1,2)$. From (13) and making use of $\Delta_1=0$ we get

$$E_2=\frac{E_1+\overline{E_1}}{2}+\Gamma\frac{\kappa_1+1}{\kappa_2+1}\cdot\frac{E_1-\overline{E_1}}{2}-i\frac{1-\Gamma}{\kappa_2+1}(E_1+\overline{E_1}\sin 2\theta_1$$

$$C_2 = \frac{C_1+\overline{C_1}}{2} + \Gamma\frac{\kappa_1+1}{\kappa_2+1}\cdot\frac{C_1-\overline{C_1}}{2} - i\frac{1-\Gamma}{\kappa_2+1}(C_1+\overline{C_1})\sin 2\theta_1$$

$$+\frac{1-\Gamma}{\kappa_2+1}[-(E_1-\overline{E_1})(1-e^{2i\theta_1}+2i\theta_1 e^{2i\theta_1})+(E_1+\overline{E_1})(1-e^{2i\theta_1}-2i\theta_1 e^{2i\theta_1})]$$

$$A_2 = \frac{A_1+\overline{A_1}}{2} + \Gamma\frac{\kappa_1+1}{\kappa_2+1}\cdot\frac{A_1-\overline{A_1}}{2} - i\frac{1-\Gamma}{\kappa_2+1}(A_1+\overline{A_1})\sin 2\theta_1 + \frac{1-\Gamma}{\kappa_2+1}\left[-\frac{C_1-\overline{C_1}}{2}\right.$$

$$\left.\cdot(1-e^{2i\theta_1}+2i\theta_1 e^{2i\theta_1})+\frac{C_1+\overline{C_1}}{2}(1-e^{2i\theta_1}-2i\theta_1 e^{2i\theta_1})+E_1\cdot 4\theta_1^2 e^{2i\theta_1}\right]$$

$$-\frac{2\mu_2}{\kappa_2+1}(\alpha_2^*-\alpha_1^*)\Delta T \tag{64}$$

it is easily observed that eqn (12c) is satisfied automatically.

Substituting (64) into (12d) and (12e), then separating the real parts from the imaginary parts, one obtains

$$a_{11}\cdot(E_1+\overline{E_1})+a_{12}\cdot(E_1-\overline{E_1})i = 0$$

$$a_{21}\cdot(E_1+\overline{E_1})+a_{22}\cdot(E_1-\overline{E_1})i = 0 \tag{65}$$

$$a_{11}\cdot(C_1+\overline{C_1})/2+a_{12}\cdot(C_1-\overline{C_1})i/2+a_{13}\cdot(E_1+\overline{E_1})+a_{14}\cdot(E_1-\overline{E_1})i$$

$$= -\frac{4\mu_2}{\kappa_2+1}(\alpha_2^*-\alpha_1^*)\Delta T(1-\cos 2\theta_2) \tag{66}$$

$$a_{21}\cdot(C_1+\overline{C_1})/2+a_{22}\cdot(C_1-\overline{C_1})i/2+a_{23}\cdot(E_1+\overline{E_1})+a_{24}\cdot(E_1-\overline{E_1})i$$

$$= -\frac{4\mu_2}{\kappa_2+1}(\alpha_2^*-\alpha_1^*)\Delta T\sin 2\theta_2$$

where

$$a_{13} = -2\frac{1-\Gamma}{\kappa_2+1}[1-\cos 2\theta_1+(\theta_1+\theta_2)\sin 2\theta_1-2\theta_2\sin 2\theta_1\cos 2\theta_1+2\theta_2^2\cos 2\theta_1$$

$$-2\theta_2(\theta_1+\theta_2)\cos 4\theta_1]+\Gamma\frac{\kappa_1+1}{\kappa_2+1}\cdot 2(\theta_1^2-\theta_2^2)\cos 2\theta_1$$

$$a_{23} = -2\frac{1-\Gamma}{\kappa_2+1}[-(\theta_1-\theta_2)\cos 2\theta_1(1-\cos 2\theta_1)+(\theta_1+\theta_2)\sin^2 2\theta_1+2(\theta_1^2-\theta_2^2)$$

$$\cdot\sin 2\theta_1+4\theta_2(\theta_1+\theta_2)\sin 2\theta_1\cos 2\theta_1]-\Gamma\frac{\kappa_1+1}{\kappa_2+1}\cdot 2(\theta_1^2-\theta_2^2)\sin 2\theta_1$$

$$a_{14} = 2\frac{1-\Gamma}{\kappa_2+1}[(\theta_1-\theta_2)\cos 2\theta_1\cdot(1-\cos 2\theta_1)-(\theta_1+\theta_2)\sin^2 2\theta_1$$

$$+4\theta_1\theta_2\sin 2\theta_1\cos 2\theta_1]+\Gamma\frac{\kappa_1+1}{\kappa_2+1}\cdot 2(\theta_1^2-\theta_2^2\sin 2\theta_1$$

$$a_{24}=2\frac{1-\Gamma}{\kappa_2+1}[-(1-\cos 2\theta_1)+(\theta_1+\theta_2)\sin 2\theta_1-2\theta_2\sin 2\theta_1\cos 2\theta_1$$

$$+2\theta_1\theta_2\cos 4\theta_1-2\theta_1^2(1-\cos 2\theta_1)]+\Gamma\frac{\kappa_1+1}{\kappa_2+1}\cdot 2(\theta_1^2-\theta_2^2)\cos 2\theta_1 \tag{67}$$

Applying the method similar to that in 3.2 to solve the set of equations (65)–(66), and noticing that when $\Delta_1=\Delta_2=0$, the coefficients a_{11} and a_{12} do not vanish simultaneously, we get

$$E_1+\overline{E_1}=\frac{1}{\Delta_3}\cdot\frac{4\mu_2}{\kappa_2+1}(\alpha_2^*-\alpha_1^*)\Delta T\cdot[a_{12}\sin 2\theta_2-a_{22}(1-\cos 2\theta_2)]$$

$$E_1-\overline{E_1}=i\frac{1}{\Delta_3}\cdot\frac{4\mu_2}{\kappa_2+1}(\alpha_2^*-\alpha_1^*)\Delta T\cdot[a_{11}\sin 2\theta_2-a_{21}(1-\cos 2\theta_2)]$$

$$\frac{C_1+\overline{C_1}}{2}=k_0a_{12}+\frac{1}{\Delta_3}\cdot\frac{4\mu_2}{\kappa_2+1}(\alpha_2^*-\alpha_1^*)\Delta T\cdot[a_{14}\sin 2\theta_2-a_{24}(1-\cos 2\theta_2)]$$

$$\frac{C_1-\overline{C_1}}{2}=ik_0a_{11}+i\frac{1}{\Delta_3}\cdot\frac{4\mu_2}{\kappa_2+1}(\alpha_2^*-\alpha_1^*)\Delta T\cdot[a_{13}\sin 2\theta_2-a_{23}(1-\cos 2\theta_2)] \tag{68}$$

where

$$\Delta_3=a_{22}a_{13}-a_{12}a_{23}-a_{21}a_{14}+a_{11}a_{24}$$

$$=2\frac{1-\Gamma}{\kappa_2+1}\left\{2\frac{1-\Gamma}{\kappa_2+1}[4(1-\cos 2\theta_1)^2-2(1-\cos 2\theta_1)\sin 2\theta_1\cdot(\theta_1-\theta_2+2\theta_2\cos 2\theta_1)\right.$$

$$-4(1-\cos 2\theta_1)\cos 2\theta_1\cdot(\theta_1^2-\theta_2^2+2\theta_2^2\cos 2\theta_1)-4\theta_1\theta_2(\theta_1+\theta_2)\sin 2\theta_1\cos 2\theta_1$$

$$\left.\cdot(1-2\cos 2\theta_1)]-\Gamma\frac{\kappa_1+1}{\kappa_2+1}\cdot 2(\theta_1-\theta_2)\cdot 2(1-\cos 2\theta_1)[2(\theta_1+\theta_2)\cos 2\theta_1+\sin 2\theta_1]\right\}$$

$$=\frac{8}{(1-\alpha)^2}\left\{2\alpha^2+\alpha[\beta(\theta_1+\theta_2)+(\theta_1-\theta_2)]\sin 2\theta_1+\frac{2\alpha\beta}{\beta-\alpha}[\beta(\theta_1^2+\theta_2^2)\right.$$

$$\left.+(\theta_1^2-\theta_2^2)]+2\beta(\beta+\alpha)\theta_1\theta_2(\theta_1+\theta_2)\sin 2\theta_1\right\} \tag{69}$$

k_0 is an arbitrary real constant.

The complex constant A_1 can be arbitrary, E_2, C_2 and A_2 can be derived from (64), then from (11) F_j, D_j and B_j can be achieved. Finally, from (8) the solution for $\Delta_1=\Delta_2=0$ is

$$\left.\begin{aligned}\varphi_1(z)&=A_1z+C_1z\ln z+E_1z\ln^2 z\\ \psi_1(z)&=B_1z+D_1z\ln z+F_1z\ln^2 z\end{aligned}\right\}\quad\text{and}\quad\begin{aligned}\varphi_2(z)&=A_2z+C_2z\ln z+E_2z\ln^2 z\\ \psi_2(z)&=B_2z+D_2z\ln z+F_2z\ln^2 z\end{aligned} \tag{70}$$

in which the terms related to the arbitrary complex constant A_1 or the arbitrary real constant k_0 represent a homogeneous solution. Equation (70) can be called the secondary paradox solution.

Substituting (70) into (4) and (5) yields

$$\begin{aligned}\sigma_{jr}-i\tau_{jr\theta} = {} & [1+e^{2i(\theta-\theta_j)}]\cdot\{(E_j+\overline{E_j})[\ln^2 r+2i\theta\ln r-\theta^2+2(\ln r+i\theta)] \\ & +(C_j+\overline{C_j}+2E_j-4i\theta_j\overline{E_j})\cdot(\ln r+i\theta+1)+(A_j+\overline{A_j}+C_j \\ & -2i\theta_j\overline{C_j}-4\theta_j^2\overline{E_j})\}-4i(\theta-\theta_j)\overline{E_j}(\ln r+1-i\theta_j)-2i(\theta-\theta_j)\overline{C_j} \\ & -4E_j(\ln r+i\theta+1)-2C_j \\ \sigma_{j\theta}+i\tau_{jr\theta} = {} & [1-e^{2i(\theta-\theta_j)}]\cdot\{(E_j+\overline{E_j})[\ln^2 r+2i\theta\ln r-\theta^2+2(\ln r+i\theta)] \\ & +(C_j+\overline{C_j}+2E_j-4i\theta_j\overline{E_j})\cdot\ln r+i\theta+1)+(A_j+\overline{A_j}+C_j \\ & -2i\theta_j\overline{C_j}-4\theta_j^2\overline{E_j})\}-4i(\theta-\theta_j)\overline{E_j}(\ln r+1-i\theta_j)-2i(\theta-\theta_j)\overline{C_j} \quad (j=1,2)\end{aligned} \tag{71}$$

it is observed that the singularity of $\ln^2 r$ appears in thermal residual stresses when $E_j+\overline{E_j}\neq 0$.

Numerical computations demonstrate that the value of Δ_3, the denominator of the solution (70), does not vanish when $\Delta_1=\Delta_2=0$, thereby the paradox does not exist for the secondary paradox solution (70).

4.2. $\sin\theta_1\neq 0$, $\sin\theta_2=0$

In this case $\theta_1\neq\pi$, $\theta_2=-\pi$. Taking $C_j=D_j=E_j=F_j=G_j=H_j=M_j=N_j=0$, $(j=1,2)$, solving eqns (12) and (13), and from eqns (11) one obtains

$$\begin{aligned}&A_1+\overline{A_1}=0,\quad A_2=\Gamma\frac{\kappa_1+1}{\kappa_2+1}A_1-\frac{2\mu_2}{\kappa_2+1}(\alpha_2^*-\alpha_1^*)\Delta T \\ &B_1=0,\quad B_2=\frac{4\mu_2}{\kappa_2+1}(\alpha_2^*-\alpha_1^*)\Delta T\end{aligned} \tag{72}$$

thus from (8) the solution is

$$\left.\begin{aligned}\varphi_1(z)&=A_1z\\ \psi_1(z)&=0\end{aligned}\right\}\quad\text{and}\quad\left.\begin{aligned}\varphi_2(z)&=A_2z\\ \psi_2(z)&=B_2z\end{aligned}\right\} \tag{73}$$

where A_1 is an arbitrary imaginary constant.

It is evident that the thermal residual stresses corresponding to the particular solution (73) vanish in material 1 and on the interface.

4.3. $\sin\theta_1=0$, $\sin\theta_2\neq 0$

In this case $\theta_1=\pi$, $\theta_2\neq-\pi$. Taking $C_j=D_j=E_j=F_j=G_j=H_j=M_j=N_j=0$, $(j=1,2)$, solving eqns (12) and (13), and from eqns (11) one obtains

$$\frac{A_1+\overline{A_1}}{2}=\frac{\kappa_2+1}{\Gamma(\kappa_1+1)}\cdot\frac{2\mu_2}{\kappa_2+1}(\alpha_2^*-\alpha_1^*)\Delta T,\quad A_2=\Gamma\frac{\kappa_1+1}{\kappa_2+1}\cdot\frac{A_1-\overline{A_1}}{2}$$

$$B_1 = -\frac{\kappa_2+1}{\Gamma(\kappa_1+1)}\cdot\frac{4\mu_2}{\kappa_2+1}(\alpha_2^*-\alpha_1^*)\Delta T, \quad B_2 = 0 \tag{74}$$

thus from (8) the solution is

$$\left.\begin{aligned}\varphi_1(z) &= \frac{A_1-\overline{A_1}}{2}z+\frac{A_1+\overline{A_1}}{2}\bar{z}\\ \psi_1(z) &= B_1 z\end{aligned}\right\} \quad \text{and} \quad \left.\begin{aligned}\varphi_2(z) &= A_2 z\\ \psi_2(z) &= 0\end{aligned}\right\} \tag{75}$$

where $A_1-\overline{A_1}/2$ is an arbitrary imaginary constant.

It is evident that the thermal residual stresses corresponding to the particular solution (75) vanish in material 2 and on the interface. Actually, if we exchange the signs 1 and 2 of the two dissimilar materials and reverse the rotation direction of θ, the situation of 4.3. is consistent with that of 4.2.

4.4. $\sin\theta_1 = 0$ and $\sin\theta_2 = 0$

In this case $\theta_1 = \pi$, $\theta_2 = -\pi$, hence the model becomes the interfacial crack. Taking $C_j = D_j = E_j = F_j = G_j = H_j = M_j = N_j = 0$, $(j = 1,2)$, solving eqns (12) and (13), and from eqns (11) one obtains

$$\begin{aligned}A_2 &= \Gamma\frac{\kappa_1+1}{\kappa_2+1}A_1-\frac{2\mu_2}{\kappa_2+1}(\alpha_2^*-\alpha_1^*)\Delta T\\ B_1 &= -(A_1+\overline{A_1}), \quad B_2 = -\left[\Gamma\frac{\kappa_1+1}{\kappa_2+1}(A_1+\overline{A_1})-\frac{4\mu_2}{\kappa_2+1}(\alpha_2^*-\alpha_1^*)\Delta T\right]\end{aligned} \tag{76}$$

thus from (8) the solution is

$$\left.\begin{aligned}\varphi_1(z) &= A_1 z\\ \psi_1(z) &= B_1 z\end{aligned}\right\} \quad \text{and} \quad \left.\begin{aligned}\varphi_2(z) &= A_2 z\\ \psi_2(z) &= B_2 z\end{aligned}\right\} \tag{77}$$

where A_1 is an arbitrary complex constant, its related terms represent a homogeneous solution.

Substituting (77) into (5) yields

$$\sigma_{j\theta}+i\tau_{jr\theta} = (A_j+\overline{A_j})(1-e^{2i\theta}) \quad (j = 1,2) \tag{78}$$

It is observed that the thermal residual stresses corresponding to the particular solution (77) vanish on the interface.

Up till now, all the cases of $\Delta = 0$ have been studied and the corresponding particular solutions presented.

5. Conclusion

In this paper, all the cases of the thermal residual stress field near the apex in dissimilar materials bonded with two arbitrary angles are researched theoretically, and the corresponding particular solutions provided. The main results are as follows:

(1) Thermal residual stresses are proportional to the temperature difference ΔT and the difference of the thermal expansion coefficients $(\alpha_2-\alpha_1)$ for plane stress or $[(1+\nu_2)\alpha_2-(1+\nu_1)\alpha_1]$ for plane strain. Moreover, thermal residual stresses are dependent on the geometry angles θ_1, θ_2 and the Dundurs parameters α, β, and $\mu_2/(\kappa_2+1)$ or $\mu_1/(\kappa_1+1)$. Especially, when θ_1, θ_2 and α, β satisfy the definite relations, logarithmic singularities develop.

(2) For the case of $\sin\theta_1\cdot\sin\theta_2\cdot\sin(\theta_1-\theta_2)\neq 0$, when $\Theta_1=0$, i.e., the expression (21) vanishes, the singularity of $\ln r$ appears in thermal residual stresses; when $\Theta_1=\Theta_2=0$, i.e., the expressions (21) and (28) vanish simultaneously, the singularities of $\ln^2 r$ and $\ln r$ appear; furthermore, when $\Theta_1=\Theta_2=\Theta_3=0$, i.e., the expressions (21), (28) and (40) vanish simultaneously, the singularities of $\ln^3 r$, $\ln^2 r$ and $\ln r$ appear.

(3) For the case of $\sin\theta_1\cdot\sin\theta_2\cdot\sin(\theta_1-\theta_2)=0$, logarithmic singularities also probably develop in thermal residual stresses. Under the circumstances of $\theta_1-\theta_2=\pi$ or 2π, and $\theta_1\neq\pi$, $\theta_2\neq-\pi$, when $\Delta_1=0$, i.e., the expression (54) vanishes, the singularity of $\ln r$ appears; moreover, when $\Delta_1=\Delta_2=0$, i.e., the expressions (54) and (61) vanish simultaneously, the singularities of $\ln^2 r$ and $\ln r$ appear. Especially, it should be pointed out that if

$$a_{12}\sin 2\theta_1-a_{22}(1-\cos 2\theta_1)=0 \quad \text{i.e.,} \quad \tan\theta_1=\theta_1-\frac{1+\alpha}{2\alpha}(\theta_1-\theta_2)$$

then the terms with the highest order of the power of $\ln r$, i.e., the terms of $(\ln r)^1$ for $\Delta_1=0$, $\Delta_2\neq 0$ and the terms of $(\ln r)^2$ for $\Delta_1=\Delta_2=0$, will vanish from the expressions of stresses, and the highest orders of the power of $\ln r$ then become $(\ln r)^0$ and $(\ln r)^1$, respectively.

(4) The logarithmic singularities of thermal residual stresses at the apex in bonded dissimilar materials have intimate relations with paradox problems. It is shown that by constructing the particular solution sequences in the form of complex functions about $(\ln z)^n$ $(n=1,2,\ldots)$, and taking many enough terms in the sequences to make linear combinations for all the complex potentials, we can solve multiple paradox problems succinctly and effectively, the resolution of the problem here is just an outstanding example for the applications of the above method.

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